DIMENSION AND PROJECTIONS IN NORMED SPACES AND RIEMANNIAN MANIFOLDS

Inauguraldissertation
der Philosophisch-naturwissenschaftlichen Fakultät
der Universität Bern

vorgelegt von

ANNINA ISELI
von Thunstetten

Leiter der Arbeit
Prof. Dr. Zoltán Balogh
Mathematisches Institut
der Universität Bern

DIMENSION AND PROJECTIONS IN NORMED SPACES AND RIEMANNIAN MANIFOLDS

Inauguraldissertation
der Philosophisch-naturwissenschaftlichen Fakultät
der Universität Bern

vorgelegt von

ANNINA ISELI

von Thunstetten

Leiter der Arbeit

Prof. Dr. Zoltán Balogh

Mathematisches Institut
der Universität Bern

Von der Philosophisch-naturwissenschaftlichen Fakultät angenommen.

Bern, 27. April 2018

Der Dekan:
Prof. Dr. Gilberto Colangelo

PREFACE

This thesis contains the results achieved during my PhD studies under the supervision of Prof. Dr. Zoltán Balogh at the Mathematical Institute of the University of Bern. During these studies, I have been supported by the Swiss National Science Foundation (project numbers 2000020_146477 and 2000020_165507).

Some of the results presented in this thesis have been published in [3] and [4]. These publications contain parts of Section 5.3 and Section 6.1, respectively. A submission comprising the results of Chapter 4, Chapter 5 and Section 6.2 is in preparation. Aside from the main project of the thesis on dimension and projections, I have worked on the topic of iterated function system quasiarcs in collaboration with Dr. Kevin Wildrick. The results that emerged from this collaboration are published in [21]. They are not included in this thesis.

Annina Iseli
Bern, 27. April 2018

ACKNOWLEDGMENTS

I am very grateful for having been surrounded by so many inspiring people and passionate mathematicians during my PhD studies.

First of all, I wish to thank my advisor Zoltán Balogh for his guidance. I am sincerely grateful to Zoltán for sharing his mathematical expertise, for always finding time for my questions, and for honestly caring about my work.
Moreover, I wish to express my gratitude to Kevin Wildrick for his encouragement and persistent enthusiasm, for numerous inspiring conversations, and for our enjoyable collaboration. The external referee for this thesis is Jeremy Tyson. I would like to thank him, as well as Zoltán and Kevin, for their careful reading and for providing useful remarks on earlier versions of this thesis.

In the past two years I had the chance to spend several short research stays abroad, therefore I am indebted to Mario Bonk, Luca Rizzi, and Kevin Wildrick for their academic hospitality. Further, I wish to thank Sebastian Baader, Katrin Fässler, Lukas Lewark, Petr Siegl, Stefan Wenger, and Roger Züst for enlightening discussions and for sharing their insight into mathematics and academic life. I would like to extend my thanks to the members of the Mathematical Institute in Bern for countless friendly encounters, mathematical discussions, IT support, fun lunch breaks, and the stimulating work atmosphere.

My time at the University of Bern wouldn't have been as cheerful and encouraging without my mathematical fellows and long-time friends Johannes Josi, Livio Liechti, and Luca Studer, as well as my good friend Karla Koller. My warmest thanks for their inspiration, support, and much appreciated sense of humor.

Finally, I wish to express my deep gratitude to my family Uschi, Urs, Lisa, and Sebastian for their continual warmth, love, and support. In particular, I am incredibly grateful to my parents Uschi and Urs for constantly encouraging me to be curious, considerate, and confident.

CONTENTS

Chapter 1

INTRODUCTION

In this thesis, we consider families of projections in metric spaces and study the change of Hausdorff measure and Hausdorff dimension of Borel sets under these projections. This chapter introduces the reader to the topic of dimension and projections and its history which is closely tied to the pioneering works of A. S. Besicovitch [7] and J. Marstrand [27].

A FUNDAMENTAL QUESTION

Let S be a non-degenerate line segment in $\mathbb{R}^2$ and consider its image under the orthogonal projection onto a one-dimensional linear subspace (line) L in $\mathbb{R}^2$. Obviously, this image is itself a non-degenerate line segment except for the case when S is orthogonal to L. This generalizes easily to the case when S is a topological arc in $\mathbb{R}^2$. Namely, the image of a topological arc S under the orthogonal projection onto L is a non-degenerate line segment for all except (at most) one line L. In particular, we may conclude that for almost every line L in $\mathbb{R}^2$ the image of a given topological arc in $\mathbb{R}^2$ under the orthogonal projection onto L is a set of positive Hausdorff 1-measure and, in particular, it is a set of dimension 1. We aim to generalize this observation to a larger class of subsets of $\mathbb{R}^2$.

Denote the orthogonal projection of $\mathbb{R}^2$ onto a line L by $P_L : \mathbb{R}^2 \to L$. Further, we denote the Hausdorff dimension of a set $A \subset \mathbb{R}^2$ by $\dim A$ and the Hausdorff 1-measure of A by $\mathscr{H}^1(A)$. The projections $P_L : \mathbb{R}^2 \to L$ are 1-Lipschitz mappings which do not increase the distance of points. Lipschitz mappings do not increase the Hausdorff measure or dimension of sets. This yields that $\dim P_L(A) \leq \dim A$ for all $A \subset \mathbb{R}^2$ and for all lines L. Furthermore, by monotonicity of the Hausdorff dimension and the fact that $P_L(A) \subset L$ and $\dim L = 1$ it follows that $\dim P_L(A) \leq 1$. Thus, we have two upper bounds for the $\dim P_L(A)$ that hold for all $A \subset \mathbb{R}^2$ and all lines L. By considering the case that A is a non-degenerate line segment we see that these estimates cannot be improved.

Optimal lower bounds on dim $P_L(A)$ are much more difficult to achieve. However, the simple example where A is a topological arc reveals a surprising amount about the general case. In 1939, Besicovitch [7] studied the behavior of 1-rectifiable sets $A \subset \mathbb{R}^2$ under orthogonal projections. Heuristically, a set is 1-rectifiable if it admits a curve-like local structure. Besicovitch's result states that if a set $A \subset \mathbb{R}^2$ is 1-rectifiable with $0 < \mathscr{H}^1(A) < \infty$, then the measure $\mathscr{H}^1(P_L(A))$ is positive (and thus $\dim P_L(A) = 1$) for almost every line L. Moreover, he proved that (roughly speaking) also the converse holds.

In particular, if a set $A \subset \mathbb{R}^2$ satisfying $\mathscr{H}^1(A) < \infty$ fails to have a curve-like local structure, then $\mathscr{H}^1(P_L(A)) = 0$ for a set of lines of positive measure. Recall that by a "line" we mean a linear subspace of $\mathbb{R}^2$. By identifying every line L in $\mathbb{R}^2$ with the counterclockwise angle from the positive x-axis to L, the term "for almost every line" can be understood with respect to the Hausdorff 1-measures on the set of angles $[0, \pi)$.

Besicovitch's result only partially answers our original question. Though we know that for sets $A \subset \mathbb{R}^2$ that are not 1-rectifiable, $\mathscr{H}^1(P_L(A)) = 0$ for a large set of lines L, this does not yield any information on the dimension of $P_L(A)$ for lines L in this set. It is easy to construct compact sets of dimension equal to 1 that fail to be 1-rectifiable. The most well-known such example is the four-corner Cantor set; see Example 15.2 in [30], and Chapter 10 in [32]. In addition, every set $A \subset \mathbb{R}^2$ that has dimension greater than 1 fails to be 1-rectifiable. Therefore, Besicovitch's result does not yield any information about the distortion of this type of sets either. More insight can be gained from a result due to Marstrand [27] from 1954. Marstrand's theorem states that the trivial upper bounds we deduced above in fact represent the generic dimension of the images of Borel sets under orthogonal projections in $\mathbb{R}^2$. Namely, it states that, for all Borel sets $A \subseteq \mathbb{R}^2$ and for almost every line L,

$$\dim P_L(A) = \min\{1, \dim A\}. \tag{1.1}$$

A BRIEF HISTORY OF PROJECTION THEOREMS

Marstrand's result marked the start of a sequence of numerous strong results in the same spirit that are often referred to as Marstrand-type projection theorems. In 1968, Kaufman [24] reproved and improved (1.1) by introducing potential theoretic methods for the study of the dimension of sets. In particular, he proved that for all Borel sets $A \subseteq \mathbb{R}^2$ with $\dim A \leq 1$,

$$\dim(\{L : \dim P_L(A) < \dim A\}) \leq \dim A. \tag{1.2}$$

In 1975, Mattila [29] adapted Kaufman's potential theoretical approach and thereby generalized (1.1) and (1.2) to include families of projections onto m-dimensional linear subspaces (m-planes) of $\mathbb{R}^n$. In particular, the higher dimensional version of (1.1) states that, for all Borel sets $A \subseteq \mathbb{R}^n$ with $\dim A \leq m < n$, and almost every m-plane V in $\mathbb{R}^n$, the image of A under the orthogonal projection $P_V : \mathbb{R}^n \to V$ is a set of dimension $\dim A$. In order to formally make sense of Mattila's result notice that the family of m-planes in $\mathbb{R}^n$ can be viewed as an $(n-m)m$-dimensional smooth manifold called the Grassmannian $G(n, m)$ that is equipped with a natural $(n-m)m$-dimensional measure induced by the action of $O(n)$ on $G(n, m)$; see Section 2.3. Besicovitch's result on the interplay of rectifiability and projections has also been generalized to higher dimensions; see Theorem 3.7. This is due to Federer [17].

In 1982, Falconer [13] was the first to apply Fourier analytic methods to problems in dimension and projections. He reproved some of the previous results and established stronger versions of them. In particular, one of his results states that for all Borel sets $A \subseteq \mathbb{R}^2$ with $\dim A > 1$,

$$\dim(\{L : \dim P_L(A) < 1\}) \leq 2 - \dim A. \tag{1.3}$$

An analogous statement holds for the family of projections onto m-planes in $\mathbb{R}^n$.

Many of these results have proven to be sharp. Given a Borel set $A \subseteq \mathbb{R}^2$ we will often call the set of lines for which the orthogonal projection does not satisfy the generic property of the respective Marstrand-type result the exceptional set of lines. For all parameters $0 < s \leq 2$, there exists a Borel set $A \subseteq \mathbb{R}^2$ of dimension s for which the exceptional set E of lines is a set of dimension s; see [25]. This proves the sharpness of (1.2). Similar result are known for the analog of (1.2) in higher dimensions, as well as for (1.3); see [13]. Sharpness for the higher dimensional version of (1.3) is open.

The constructions of sets $A \subseteq \mathbb{R}^n$ that reveal the sharpness of Marstrand-type projection theorems are very specific and in general do not yield any information on the structure of exceptional sets in general. The study of the structure and size of exceptional sets began in 2008 with the work of Järvenpää et. al. [22] on one-dimensional families of lines in $\mathbb{R}^3$. For a detailed account on the latest progress in this area see the recent works [16], [33], [9], and the references therein.

Marstrand-type projection theorems have also been studied for notions of dimension other than the Hausdorff dimension; see [23], [14], [15], and references therein. Moreover, the expository articles [31] and [28] are highly recommended.

THE METHOD OF TRANSVERSALITY

As mentioned above, the methods that Falconer employed in order to reprove and extend earlier results are heavily based on Fourier analysis. For some of his results no proof without Fourier analysis is known. Falconer's Fourier analytic methods for geometric measure theory have been further developed by numerous mathematicians. In particular, Peres and Schlag [34] established a very general theorem about families of abstract projections from compact metric spaces to Euclidean space and their impact on the Sobolev-dimension of Borel measures. While the main applications of their results concern Bernoulli convolutions, all the classical Marstrand-type projection theorems stated above can be deduced as corollaries from their main result.

Even though the Fourier methods by Peres and Schlag differ substantially from Kaufman's and Mattila's potential theoretic approach, there is a common ground: the notion of transversality. Requiring a family of (abstract) projections to be transversal guarantees that there are very few pairs of points such that the distance between the image of the points under a projection (onto some line resp. m-plane) is very small compared to the distance between the two points themselves. However, the way this rareness is controlled differs substantially between the potential theoretic and the Fourier method. In potential theoretic proofs of Marstrand-type projection theorems, one is concerned with the condition of metric transversality (Definition 3.2). This condition imposes an upper bound on the Grassmannian measure of the set of lines (resp. planes) in $\mathbb{R}^2$ (resp. $\mathbb{R}^n$) for which the distance between the image of two distinct points is comparatively small. On the other hand, the Fourier analytic proof of projection theorems for abstract projections works with the notion of differentiable transversality. Differentiable transversality requires that if the ratio Φ of the distance of two projected points and the distance of the points themselves is small, then Φ grows fast when the projection parameter is altered. A precise definition can be found in Section 3.2.2. Similar notions of transversality have been studied for example by Solomyak in [35]. Moreover, Hovila et. al. [20] have shown that the Besicovitch-Federer projection theorem is also a direct consequence of a sufficiently strong version of differentiable transversality.

Moreover, Marstrand-type results have been successfully studied in non-Euclidean spaces. Balogh et. al. [1] established counterparts for Marstrand's projection theorem for the family of projections onto horizontal lines and the family of projections onto vertical planes in the first Heisenberg group. Moreover, in [2] they give counterparts for these results in higher dimensional Heisenberg groups. Both these works employ methods similar to the potential theoretic methods mentioned above. Furthermore, Hovila [19] proved that the families of isotropic projections in the Heisenberg groups satisfy a version of differentiable transversality that is strong enough for many Marstrand-type projection theorems as well as the Besicovitch-Federer projection theorem to hold.

PROJECTIONS IN NORMED SPACES AND RIEMANNIAN MANIFOLDS

In this thesis, we establish Marstrand-type projection theorems for closest-point projections in sufficiently regular normed spaces as well as on Riemannian manifolds of constant curvature. Chapter 2 contains preliminaries and can be safely skipped by experts. In Chapter 3, we give the formal definitions of metric and differentiable transversality, and compare these definitions.

In Chapter 4, we establish sufficient conditions for a family of linear and surjective projections (Definition 4.1) to satisfy Marstrand-type projection theorems; see Theorem 4.2. These conditions turn out to be essentially necessary. Moreover, we consider a weaker version of differentiable transversality that we show to be equivalent to differentiable transversality for families of linear projections; see Proposition 4.7.

In Chapter 5, we consider finite dimensional normed spaces, i.e., we equip $\mathbb{R}^n$ with a strictly convex norm $\|\cdot\|$, and study the family of closest-point projections $P^{\|\cdot\|}$ onto m-planes with respect to $\|\cdot\|$. Note that by the assumption of strict convexity of $\|\cdot\|$

these closest-point projections are well-defined. If the norm $\|\cdot\|$ is sufficiently regular, then a comparison argument shows that the family of closest-point projections onto $(n-1)$-planes is a family of linear and surjective projections for which Theorem 4.2 applies; see Lemma 5.6 and Theorem 5.5. The same methods provide a Besicovitch-Federer characterization of purely unrectifiable sets in terms of closest-point projections; see Corollary 5.8. Moreover, Theorem 5.16 states that any strictly convex norm in $\mathbb{R}^2$ that barely fails the assumptions of Theorem 5.5 does not support Marstrand-type projection theorems. In the proof of Theorem 5.16 we explicitly construct a norm for which Marstrand's and Kaufman's Theorem fail. Whether or not this provides a rigorous proof of the sharpness of Theorem 5.5 depends on open problems concerning the structure of exceptional sets for Euclidean projections. Aside from these results obtained by comparison arguments, we also investigate differentiable transversality for the family of closest-point projections onto $(n-1)$-planes in $\mathbb{R}^n$ with respect to strictly convex norms. Theorem 5.9 proves that under slightly stronger regularity assumptions for $\|\cdot\|$, the according family of closest-point projections satisfies differentiable transversality. This is of particular interest in light of a recent result due to Bate, Csörnyei, and Wilson [5] which states that differentiable transversality fails for closest-point projections in infinite dimensional Banach spaces. Finally, Corollary 5.14 reveals that establishing differentiable transversality in order to prove Marstrand-type results for closest-point projections in finite dimensional normed spaces is in general not efficient.

In Chapter 6, we study the same questions for orthogonal projections along geodesics in Riemannian manifolds. Fix a base point p in a simply connected Riemannian manifold M of constant sectional curvature. We call a submanifold V of M a geodesic m-plane if V is the image of a linear m-plane under the exponential map at p. Then, all geodesic m-planes are geodesically convex subspaces of M. Hence, the projections onto m-planes are globally defined for manifolds of constant negative sectional curvature; and they are defined in an open ball of radius r and center p for manifolds with constant positive sectional curvature less than or equal to $\frac{1}{r^2}$. In Theorem 6.1 (resp. 6.4) we establish differentiable transversality for the family of orthogonal projections onto geodesic lines in the hyperbolic two-plane $\mathbb{H}^2$ (resp. geodesic segments in an open half-sphere of S^2). Thereby we prove Marstrand-type projection theorems as well as the Besicovitch-Federer projection theorem in these settings; see Corollary 6.2 and 6.5. López et. al. [26] have generalized parts of these results to surfaces of negative curvature by a case study. Theorem 6.7 states that the Marstrand-type results known to hold for projections onto lines in the hyperbolic plane generalize projections onto m-planes in hyperbolic n-space $\mathbb{H}^n$. By consideration of the Klein model for hyperbolic space one may view the family of orthogonal projections onto lines in $\mathbb{H}^n$ as a family of linear projections. Hence, Marstrand-type projection theorems in $\mathbb{H}^n$ (Theorem 6.7) can be deduced from Theorem 4.2. Furthermore, we establish differentiable transversality for the family of orthogonal projections onto lines in $\mathbb{H}^n$ by studying the transition from the Poincaré model of $\mathbb{H}^n$ to the Klein model.

Chapter 2

PRELIMINARIES

The main part of the material presented in this chapter can be found in [30]. We also recommend [12], [18], and [11]. Experts may safely skip this chapter.

Throughout this thesis, n and m will denote positive integers with $n > m$.

2.1 MEASURES ON METRIC SPACES

Let (X, d) be a metric space and μ a measure on X. We say that a property (P) holds for μ-almost every $x \in X$ (or short, for μ-a.e. $x \in X$) if there exists a set $E \subset X$ with $\mu(E) = 0$ and all $x \in X \backslash E$ have the property (P).

The measure μ is called a Borel measure if all Borel sets in (X, d) are μ-measurable. It is called Borel regular if, in addition, for all sets $A \subseteq X$ there exists a Borel set $B \subseteq X$ such that $A \subseteq B$ and $\mu(B \backslash A) = 0$. A Borel measure is called locally finite if compact sets have finite measure. Furthermore, a measure μ on X is called a Radon measure if it is a locally finite Borel measure that is inner and outer regular, i.e.,

- $\mu(U) = \sup\{\mu(K) : K \subset U,\ K \text{ compact}\}$, for all $U \subseteq X$ open,
- $\mu(A) = \inf\{\mu(V) : A \subseteq V,\ V \subseteq X \text{ open}\}$, for all $A \subset X$.

The support of a measure μ on (X, d) is the smallest closed set $K \subseteq X$ for which $\mu(X \backslash K) = 0$. We denote the support of μ by $\operatorname{spt} \mu$.

When (X, d) is $\mathbb{R}^n$ equipped with the Euclidean metric, then

- a measure μ on $\mathbb{R}^n$ is a Radon measure if and only if it is a locally finite Borel regular measure,
- for every Borel measure μ on $\mathbb{R}^n$, there exists a Borel regular measure $\mu^\star$ such that $\mu(A) = \mu^\star(A)$ for all μ-measurable sets $A \subseteq X$.

For $A \subseteq \mathbb{R}^2$, we denote by $\mathscr{M}(A)$ the set of all non-trivial finite Borel measures μ with compact support contained in A. Notice that by the two facts above, in many applications we may assume without loss of generality that the measures in $\mathscr{M}(A)$ are Borel regular and thus Radon measures.

Let (X, d_X) and (Y, d_Y) be metric spaces, μ a measure on X, and $f : (X, d_X) \to (Y, d_Y)$

a mapping. Then, the push-forward of μ by f is a measure on Y defined by

$$f_\sharp\mu(A) := \mu(f^{-1}(A))$$

for all $A \subseteq Y$. In case μ is a Borel measure and f is a Borel function, then $f_\sharp\mu$ is a Borel measure. Thus, in particular, if $f : (X, d_X) \to (Y, d_Y)$ is continuous and μ is locally finite Borel measure with compact support in A, for some $A \subseteq X$, then $f_\sharp\mu$ is a locally finite Borel measure with compact support in $f(A)$.

Moreover, a measure μ on (X, d) is called absolutely continuous with respect to another measure ν on (X, d), if whenever $\nu(A) = 0$ for some $A \subseteq X$, then also $\mu(A) = 0$. We will mostly be interested in whether or not certain measures on $\mathbb{R}^n$ are absolutely continuous with respect the Lebesgue measure $\mathscr{L}^n$ or some s-dimensional Hausdorff measure on $\mathbb{R}^n$ (which we formally define below).

Finally, let μ and ν be measures on sets X and Y, respectively, and consider $f : (X, \mu) \to (Y, \nu)$. We say that that f has the Lusin property if whenever $\mu(A) = 0$ for some $A \subseteq X$, then $\nu(f(A)) = 0$. We say that f has the inverse Lusin property, if, whenever $\nu(B) = 0$ for some $B \subseteq Y$, then $\mu(f^{-1}(B)) = 0$. Notice that f having the inverse Lusin property is equivalent to $f_\sharp\mu$ being absolutely continuous with respect to ν. In case f is invertible, then f has the inverse Lusin property if and only if f^{-1} has the Lusin property.

2.2 HAUSDORFF MEASURE AND DIMENSION

Let (X, d) be a metric space and for a set $A \subseteq X$ we denote by $\operatorname{diam} A$ the diameter of A with respect to d. The Hausdorff s-measure on (X, d), denoted by $\mathscr{H}^s$, is defined as follows. For a set $A \subseteq X$ and a parameter $s > 0$,

$$\mathscr{H}^s(A) := \sup_{\delta > 0} \mathscr{H}^s_\delta(A) = \lim_{\delta \to 0} \mathscr{H}^s_\delta(A),$$

where

$$\mathscr{H}^s_\delta(A) := \inf \left\{ \sum_{i=1}^{N} (\operatorname{diam} A_i)^s : A_i \subset \mathbb{R}^n \text{ open},\ \operatorname{diam} A_i < \delta,\ A \subseteq \bigcup_{i \in \mathbb{N}} A_i, \right\}.$$

In case $\mathscr{H}^s(A) = 0$ for all $s > 0$, we say that the Hausdorff dimension of A (with respect to d), denoted by $\dim A$, equals ∞. On the other hand, if $\mathscr{H}^s(A) = \infty$ for all $s > 0$, then $\dim(A) = 0$. One can check that for a given $\mathscr{H}^s$-measurable Hausdorff dimension neither 0 nor ∞, there exists a unique $s_0 > 0$ such that $\mathscr{H}^s(A) = \infty$ for all $s < s_0$ and $\mathscr{H}^t(A) = 0$ for all $t > s_0$. In this case, we call s_0 the Hausdorff dimension of A with respect to d denoted by $\dim(A)$. We can thus write

$$\dim(A) = \inf\{s > 0 : \mathscr{H}^s(A) = 0\} = \inf\{s > 0 : \mathscr{H}^s(A) < \infty\}. \tag{2.1}$$

A mapping $f : (X, d_X) \to (Y, d_Y)$ is called L-Lipschitz ($L > 0$), if for all $x, x' \in A$,

$$d_Y(f(x), f(x')) \leq L\, d_X(x, x').$$

Moreover, f is called L-bi-Lipschitz, if for all $x, x' \in A$,

$$\frac{1}{L} d_X(x, x') \leq d_Y(f(x), f(x')) \leq L\, d_X(x, x').$$

It is easy to check that if f is L-Lipschitz then $\mathscr{H}^s(f(A)) \leq L^s \mathscr{H}^s(A)$ for all $s > 0$ and $A \subseteq X$, and hence, $\dim f(A) \leq \dim(A)$. Therefore, in case that f is L-bi-Lipschitz, $\frac{1}{L^s}\mathscr{H}^s(A) \leq \mathscr{H}^s(f(A)) \leq L^s \mathscr{H}^s(A)$ and hence $\dim f(A) = \dim(A)$. This shows that all Lipschitz mappings $f : (X, d_X, \mathscr{H}^s_X) \to (Y, d_Y, \mathscr{H}^s_Y)$ have the Lusin property, and if f is bi-Lipschitz, then f in addition has the inverse Lusin property.

As we will almost always consider only metric spaces that are locally bi-Lipschitz equivalent to $\mathbb{R}^n$, we omit reference to the underlying metric in our notation for Hausdorff measure and dimension. Note that for all spaces (X, d) that are bi-Lipschitz equivalent to some Euclidean space, $\mathscr{H}^s$ is known to be an inner and outer regular measure. It follows from scaling and translation arguments that $\mathscr{H}^n = C\,\mathscr{L}^n$ on $\mathbb{R}^n$, for some constant $C = C(n) > 0$.

We may generalize the above discussion about Lipschitz mappings by considering Hölder mappings. A mapping $f : (X, d_X) \to (Y, d_Y)$ is called δ-Hölder for $\delta > 0$, if there exists a constant $M > 0$ with

$$d_Y(f(x), f(x')) \leq M\, d_X(x, x')^\delta,$$

for all $x, x' \in X$. Moreover, f is called δ-bi-Hölder, if there exists a constant $M > 0$ with

$$\frac{1}{M} d_X(x, x')^\delta \leq d_Y(f(x), f(x')) \leq M\, d_X(x, x')^\delta,$$

for all $x, x' \in X$. It follows that if $f : (X, d_X) \to (Y, d_Y)$ is δ-Hölder and $A \subseteq X$, then $\dim f(A) \leq \frac{1}{\delta}\dim(A)$. Therefore, in case $f : (X, d_X) \to (Y, d_Y)$ is δ-bi-Hölder, $\dim f(A) = \frac{1}{\delta}\dim(A)$. Moreover, note that a function $f : (X, d_X) \to (Y, d_Y)$ is L-Lipschitz if and only if it is 1-Hölder with multiplicative constant $M = L$.

2.3 THE GRASSMANNIAN OF M-PLANES

The Grassmannian $G(n, m)$ is the set of all m-dimensional linear subspaces of $\mathbb{R}^n$. We will often refer to the Grassmannian elements as m-planes (in $\mathbb{R}^n$). $G(n, m)$ is usually equipped with the metric d that is defined as follows. For $V, W \in G(n, m)$,

$$d(V, W) = \|P_V^{\mathbb{E}} - P_W^{\mathbb{E}}\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the standard operator norm for linear operators $\mathbb{R}^n \to \mathbb{R}^n$ and $P_V^{\mathbb{E}}$ denotes the orthogonal projection $\mathbb{R}^n \to V \subset \mathbb{R}^n$. With this metric, $G(n,m)$ is compact.

The group of orthogonal transformation $O(n)$ acts transitively on $G(n,m)$. Therefore, the invariant Haar measure θ_n on $O(n)$, induces a measure $\sigma_{n,m}$ on $G(n,m)$ as follows. Fix $V_0 \in G(n,m)$ and for $E \subseteq G(n,m)$ set

$$\sigma_{n,m}(E) := \theta_n(\{g \in O(n) : g(V_0) \in E\}).$$

One can check that this definition does not depend on the choice of V_0 and in fact defines a Radon probability measure on $G(n,m)$. By construction, $\sigma_{n,m}$ is invariant under the action of $O(n)$, that is, $\sigma_{n,m}(E) = \sigma_{n,m}(g(E))$, for all $g \in O(n)$ and $E \subseteq G(n,m)$.

There is a natural identification of $G(n,m)$ with $G(n,n{-}m)$. Namely, for every m-plane $V \in G(n,m)$, the orthogonal complement $V^\perp$ of V is an element of $G(n,n{-}m)$. In fact, this identification is a measure preserving isometry with respect to the metric d defined above and the measure $\sigma_{n,m}$. In particular, the measure $\sigma_{n,m}$ satisfies the following symmetry property: for all sets $E \subseteq G(n,m)$, $\sigma_{n,m}(E) = \sigma_{n,n-m}(\{V^\perp : V \in E\})$.

Furthermore, one can view $G(n,1)$ as S^{n-1} in the following sense. For every $v \in S^{n-1}$, define $L_v := \{tv : t \in \mathbb{R}\}$. Then, $L : S^{n-1} \to G(n,1)$, $v \mapsto L_v$ is a surjective mapping that is also injective up to the fact that $L_v = L_{-v}$, for all $v \in S^{n-1}$. Thus, its inverse is well-defined as a set-valued map and for every $L \in G(n,1)$ there exists a $v \in S^{n-1}$ such that $h^{-1}(L) = \{v, -v\}$. By σ^{n-1} denote the normalized surface measure on S^{n-1}, then for every $E \subseteq G(n,1)$,

$$\sigma_{n,1}(E) = \sigma^{n-1}(\{v \in S^{n-1} : L_v \in E\}).$$

If we first identify $G(n,n{-}1)$ with $G(n,1)$, then $G(n,1)$ with S^{n-1}, we obtain the following identification of $G(n,n{-}1)$ with S^{n-1}. An element $V \in G(n,n{-}1)$ is identified with the directions $w, -w \in S^{n-1}$ that are orthogonal to V. In particular, it follows that

$$\sigma_{n,n-1}(E) = \sigma^{n-1}(\{v \in S^{n-1} : L_v^\perp \in E\}). \tag{2.2}$$

Furthermore, the Grassmannian $G(n,m)$ can be viewed as a smooth manifold of dimension $(n{-}m)m$. We will now define local coordinates on $G(n,m)$. By $\mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$ denote the space of $((n{-}m)\times m)$-matrices with real entries. For every $T \in \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$ denote the entries by $t_{i,j}$, $i = 1, \dots, n{-}m$, $j = 1, \dots, m$, and we write

$$T = \begin{pmatrix} t_{1,1} & t_{1,2} & \dots & t_{1,m} \\ t_{2,1} & t_{2,2} & \dots & t_{2,m} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ t_{n-m,1} & t_{n-m,2} & \dots & t_{n-m,m} \end{pmatrix}. \tag{2.3}$$

We will sometimes identify $\mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$ with $\mathbb{R}^{(n-m)m}$ by identifying the matrix T with the vector

$$(t_{1,1}, t_{1,2}, \ldots, t_{1,m}, t_{2,1}, t_{2,2}, \ldots, t_{2,m}, \ldots, t_{n-m,1}, t_{n-m,2}, \ldots, t_{n-m,m}) \in \mathbb{R}^{(n-m)m}.$$

In case $n = 2$ and $m = 1$, $\mathrm{Mat}_{(n-m)\times m}(\mathbb{R}) = \mathbb{R}$ and we will write t for T.

Let $V_0 = \mathbb{R}^m \times \{0\}^{n-m} \in G(n,m)$. Then, a local parameterization of $G(n,m)$ near $V_0 = \mathbb{R}^m \times \{0\}^{n-m} \in G(n,m)$ is given by

$$\begin{aligned} \varphi : \mathrm{Mat}_{(n-m)\times m}(\mathbb{R}) &\to G(n,m) \\ T &\mapsto V_T := \{(w, Tw) \in \mathbb{R}^n : w \in \mathbb{R}^m\}. \end{aligned} \tag{2.4}$$

Note that if T is the zero-matrix, then $V_T = V_0$, i.e., our notation is compatible. For any other choice of V_0, we can pre-compose φ with a rotation that maps $\mathbb{R}^m \times \{0\}^{n-m}$ to V_0 and thereby obtain a local parameterization of $G(n,m)$ near this new V_0. Note that the topology induced by these local charts coincides with the topology of the metric d defined above.

For our studies of projections onto elements V of the Grassmannian $G(n,m)$, we will not only need local parameterizations of $G(n,m)$, but also orthonormal bases of the elements V in terms of the parameterization. For this, let $w_1, \ldots, w_m$ be the standard (Euclidean orthonormal) basis of $\mathbb{R}^m$, and for every $T \in \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$, define

$$v_i^T := (w_i, Tw_i).$$

Then, $v_1^T, \ldots, v_m^T$ is a basis of V_T that depends smoothly on T. In particular, for all $i = 1, \ldots, m$, we have $v_i^0 = e_i$ where $e_1, \ldots, e_n$ denotes the standard basis of $\mathbb{R}^n$. Thus, it follows that $e_i = (w_i, 0) \in \mathbb{R}^m \times \mathbb{R}^{n-m} = \mathbb{R}^n$, for $i = 1, \ldots, m$. Now, for every $T \in \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$, let

$$e_1^T, \ldots, e_m^T \tag{2.5}$$

the basis of V_T obtained by applying the Gram-Schmitt algorithm to the basis $v_1^T, \ldots, v_m^T$. This makes $e_1^T, \ldots, e_m^T$ an orthonormal basis of V_T that varies smoothly in T.

Remark 2.1. Notice that for a set $E \subseteq G(n,m)$, $\mathscr{H}_d^s(E) = 0$ where $\mathscr{H}_d^s$ denotes the Hausdorff s-measure on $G(n,m)$ with respect to the Grassmannian metric d if and only if for all smooth charts $\widetilde{\varphi} : U \to G(n,m)$ with $U \subseteq \mathbb{R}^{(n-m)m}$ open we have $\mathscr{H}^s(\varphi^{-1}(E)) = 0$. Moreover, $\sigma_{n,m}(E) = 0$, if and only if $\mathscr{H}_d^{(n-m)m}(E) = 0$. In the sequel of this thesis, we will mainly be interested in whether or not certain sets $E \subseteq G(n,m)$ are zero sets with respect to either $\sigma_{n,m}$ or some Hausdorff s-measure, and we will not care about the exact value of the measure of E. Therefore, we won't distinguish (in notation and else) between $\mathscr{H}_d^s$ on $G(n,m)$ and $\mathscr{H}^s$ in its charts.

Chapter 3

THE METHOD OF TRANSVERSALITY

3.1 POTENTIAL THEORETIC METHODS AND METRIC TRANSVERSALITY

The method for proving Marstrand-type projection results presented in this chapter is originally due to Kaufman [24], who has developed it in $\mathbb{R}^2$. It has been generalized to higher dimensions and brought to the form in which we present it here by Mattila [29]. See also Chapters 8 and 9 in [30] for a detailed account.

For each $V \in G(n,m)$, define $P_V^{\mathbb{E}} : \mathbb{R}^n \to V$ to be the orthogonal projection of $\mathbb{R}^n$ onto V. It will be useful to consider the entire family $\{P_V^{\mathbb{E}} : V \in G(n,m)\}$ of projections as a single object. To this end, we define the mapping

$$P^{\mathbb{E}} : G(n,m) \times \mathbb{R}^n \to \mathbb{R}^n \tag{3.1}$$

by $P^{\mathbb{E}}(V,x) = P_V^{\mathbb{E}}(x)$. We will often refer to the mapping $P^{\mathbb{E}}$ as the family of orthogonal projections (onto m-planes) in $\mathbb{R}^n$, or as the family of Euclidean projections.

We begin with the following theorem that summarizes the theorems mentioned in the introduction. A proof can be found in [30], Chapter 9.

Theorem 3.1. *Let $A \subseteq \mathbb{R}^n$ be a Borel set.*

(1) *If $\dim A \leq m$, then*
   (a) *$\dim(P_V^{\mathbb{E}} A) \geq \dim A$ for $\sigma_{n,m}$-a.e. $V \in G(n,m)$,*
   (b) *$\dim(\{V \in G(n,m) : \dim(P_V^{\mathbb{E}} A) < \dim A\}) \leq (n-m-1)m + \dim A$.*

(2) *If $\dim A > m$, then $\mathscr{H}^m(P_V^{\mathbb{E}} A) > 0$ for $\sigma_{n,m}$-a.e. $V \in G(n,m)$.*

Notice that by inner regularity of the Hausdorff measure, it suffices to prove Theorem 3.1 for compact sets $A \subset \mathbb{R}^n$.

Having in mind the classical definition of the Hausdorff dimension (see (2.1)), one might try to prove the Theorem 3.1 by showing that if $\mathscr{H}^s(A) > 0$ for some $s > 0$, then $\mathscr{H}^s(P_V^{\mathbb{E}}(A)) > 0$ for $\sigma n, m$-a.e. $V \in G(n,m)$. Unfortunately, this does not hold. A counterexample can be found in [30], Example 9.2. Therefore, it is a better option to work with the capacitary dimension that we shall define now. Let $A \in \mathbb{R}^n$ and by $\mathscr{M}(A)$ denote the set of all non-trivial finite Borel measures on $\mathbb{R}^n$ with compact support contained in A. For $\mu \in \mathscr{M}(A)$ and $s > 0$, define the s-energy of $\mu \in \mathscr{M}(A)$ by

$$I_s(\mu) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{|x-y|^s} \mathrm{d}\mu(x)\mathrm{d}\mu(y).$$

Notice that if $I_s(\mu) < \infty$ for some $s > 0$, then $I_{s'}(\mu) < \infty$ for all $0 < s' < s$. Moreover, we call $\mu \in \mathscr{M}(A)$ a Frostman s-measure if for all $x \in \mathbb{R}^n$ and $r > 0$,

$$\mu(B(x,r)) \leq r^s,$$

where $B(x,r)$ denotes the open ball with center x and radius r in $\mathbb{R}^n$. Now, define the capacitary dimension of A to be

$$\dim_c(A) = \sup\{s > 0 : \text{ there exists } \mu \in \mathscr{M}(A) \text{ with } I_s(\mu) < \infty\}, \tag{3.2}$$

or equivalently, see Chapter 8 in [30],

$$\dim_c(A) = \sup\{s > 0 : \text{ there exists a Frostman } s\text{-measure } \mu \text{ in } \mathscr{M}(A)\}.$$

It is straight-forward to check that then $\dim(A) \geq \dim_c(A)$ for all $A \subseteq \mathbb{R}^n$. The well-known Frostman's lemma states that for all Borel sets $A \subseteq \mathbb{R}^n$ and $s > 0$, $\mathscr{H}^s(A) > 0$ if and only if there exists a Frostman s-measure in $\mu(A)$; see Theorem 8.8 in [30]. From this one easily deduces that $\dim(A) = \dim_c(A)$ whenever A is a Borel set.

Let $A \subset \mathbb{R}^n$ be a compact set and $s > 0$ such that there exists $\mu \in \mathscr{M}(A)$ with $I_s(\mu) < \infty$. One can show that

(i) if $0 < s < m$, then $I_s((P_V^{\mathbb{E}})_\sharp\mu) < \infty$ for $\sigma_{n,m}$-a.e. $V \in G(n,m)$,

(ii) if $s > m$, then $\mathscr{H}^m((P_V^{\mathbb{E}})_\sharp\mu) > 0$ for $\sigma_{n,m}$-almost every $V \in G(n,m)$.

See Chapter 9 in [30] for the proofs. Note (1.a) and (2) from Theorem 3.1 are straight-forward consequences of the facts (i) and (ii).

Now, consider a mapping

$$P : G(n,m) \times \mathbb{R}^n \to \mathbb{R}^n$$

for which $P(V,x) \in V$ for all $V \in G(n,m)$. We will call such a mapping a family of projections (onto m-planes) in $\mathbb{R}^n$ and think of them as the family $\{P_V : V \in G(n,m)\}$ where P_V is given by $P_V(x) = P(V,x)$; compare (3.1). The properties of the family $P^{\mathbb{E}}$ that are used in the proof of facts (i) and (ii) can be axiomatized as follows.

Definition 3.2. We say that a family of projections $P : G(n,m) \times \mathbb{R}^n \to \mathbb{R}^n$ is metrically transversal if the following hold.

(a) $P : G(n,m) \times \mathbb{R}^n \to \mathbb{R}^n$ is a Borel function such that for all $V \in G(n,m)$, $P_V : \mathbb{R}^n \to V$ maps bounded sets to bounded sets,

(b) there exists a constant $C > 0$ such that for every pair of distinct points $x, y \in \mathbb{R}^n$

and every $\delta > 0$,

$$\sigma_{n,m}(\{V \in G(n,m) : |P_V x - P_V y| \leq \delta\}) \leq C\delta^m |x-y|^{-m}.$$

Notice that the regularity condition (a) implies that $(P_V)_\sharp \mu \in \mathscr{M}(P_V(A))$ for all $\mu \in \mathscr{M}(A)$ and for Borel sets $A \subseteq \mathbb{R}^n$. The theorem below follows from the proof of conclusions (1.a) and (2) of Theorem 3.1 given in Chapter 9 in [30].

Theorem 3.3. *Let $P : G(n,m) \times \mathbb{R}^n \to \mathbb{R}^n$ be a metrically transversal family of projections for which $P_V : \mathbb{R}^n \to V$ is dimension non-increasing for all $V \in G(n,m)$. Then, the following hold for all Borel sets $A \subseteq \mathbb{R}^n$.*

(1) *If $\dim A \leq m$, then $\dim(P_V A) = \dim A$ for $\sigma_{n,m}$-a.e. $V \in G(n,m)$,*

(2) *If $\dim A > m$, then $\mathscr{H}^m(P_V A) > 0$ for $\sigma_{n,m}$-a.e. $V \in G(n,m)$.*

If we drop the assumption of $P_V : \mathbb{R}^n \to V$ being dimension non-increasing for all $V \in G(n,m)$, Theorem 3.3 still holds, except that (i) becomes: If $\dim A \leq m$, then $\dim(P_V(A)) \geq \dim A$ for $\sigma_{n,m}$-a.e. $V \in G(n,m)$.

It is possible to axiomatize the conditions that are necessary for (1.b) of Theorem 3.1 to hold as well, and thereby extend Theorem 3.3 by an analog of (1.b). For this, in particular, one would have to replace condition (b) in Definition 3.2 by the following stronger condition (compare [29]):

(c) for $t = s + m(n-m-1)$ and all Frostman t-measures ν on $G(n,m)$,

$$\nu(\{V \in G(n,m) : \dim(P_V^{\mathbb{E}} A) < s\}) = 0.$$

As pointed out in the introduction there is another important theorem about dimension and projections in Euclidean space due to Besicovitch [7] and Federer [17]. This theorem relates the rectifiability of a set to the Hausdorff measure of its images under orthogonal projections. A set $A \subseteq \mathbb{R}^n$ is called m-rectifiable if there exist at most countably many Lipschitz mappings $f_i : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\mathscr{H}^m\Big(A \setminus \bigcup f_i(\mathbb{R}^m)\Big) = 0.$$

It is a simple consequence of this definition that every m-rectifiable set $A \subseteq \mathbb{R}^n$ locally is of finite $\mathscr{H}^m$-measure. If $m \geq n$, then every set $A \subseteq \mathbb{R}^n$ is m-rectifiable. Therefore, the case $m \geq n$ is not of interest for our purposes and we stick to our general assumption that $m < n$. A set $E \subseteq \mathbb{R}^n$ is called purely m-unrectifiable, if $\mathscr{H}^m(E \cap A) = 0$ for every m-rectifiable set $A \subseteq \mathbb{R}^n$.

In the introduction we briefly addressed the notion of 1-rectifiability of subsets of $\mathbb{R}^2$. Namely, we gave the heuristic definition that a subset A of $\mathbb{R}^2$ is rectifiable if it has some sort of local curve-like structure. While this is not obvious from the definition of rectifiability, the heuristic notion of a local curve-like structure can be made rigorous by introducing the notion of approximate tangent lines. This yields an equivalent definition of 1-rectifiability which in fact can be generalized to an equivalent definition of m-rectifiability by a notion of (m-dimensional) approximative tangent planes; see Chapter 16 in [30].

The following theorem is widely known as the Besicovitch-Federer projection theorem. It was proven in [7] for the case when $n = 2$ and $m = 1$, and later generalized to the statement below in [17]. For a more recent account, see [30], Theorem 18.1.

Theorem 3.4. *An $\mathscr{H}^m$-measurable set $A \subseteq \mathbb{R}^n$ with $\mathscr{H}^m(A) < \infty$ is purely m-unrectifiable if and only if $\mathscr{H}^m(P_V^{\mathbb{E}}(A)) = 0$ for $\sigma_{n,m}$-a.e. $V \in G(n,m)$.*
Equivalently, A is m-rectifiable if and only if $\mathscr{H}^m(P_V^{\mathbb{E}}(B)) > 0$ for $\sigma_{n,m}$-a.e. $V \in G(n,m)$ whenever B is an $\mathscr{H}^m$-measurable subset of A with $\mathscr{H}^m(B) > 0$.

3.2 ABSTRACT PROJECTIONS AND DIFFERENTIABLE TRANSVERSALITY

In this section, we introduce a version of a strong projection theorem due to Peres and Schlag [34]; see also Chapter 18 in [31] and the survey [32]. Their main result states that if a (sufficiently regular) family of projections satisfies some sort of differentiable transversality condition, then a set of fairly strong Marstrand-type projection theorems hold. Unlike Theorem 3.1 from the previous chapter, all the results presented in this chapter are formulated for families of abstract projections, in the sense that the target space is not embedded in the domain. The notion of a family of abstract projections will be formally defined below. Furthermore, we will recall a result due to Hovila et. al. [20] that states that differentiable transversality yields a Besicovitch-Federer type characterization of purely unrectifiable sets; see Theorem 3.14.

We begin by recalling the notion of Hölder spaces. Let U be an open subset of $\mathbb{R}^n$ and $0 < \delta \leq 1$ and $k \in \mathbb{N}_0$. We say that $f : U \to \mathbb{R}^m$ is of class $C^{k,\delta}$ if f is k-times continuously differentiable (i.e. f is of class C^k) and its partial derivatives of order k are locally δ-Hölder.
In fact, the class of $C^{k,\delta}$-mappings has many properties in common with the class of C^k-mappings. In particular, products and quotients with non-vanishing denominator of mappings of class $C^{k,\delta}$ are themselves $C^{k,\delta}$. Also, whenever f, g are of class $C^{k,\delta}$ for some $k \in \mathbb{N}$, $0 < \delta < 1$, then $f \circ g$ is of class C^{k,δ^2}. Furthermore, the following version of the inverse function theorem holds for Hölder spaces: Let $f : U \to \mathbb{R}^n$ be a mapping of class $C^{k,\delta}$ or some $k \geq 1$ and $0 < \delta < 1$ where $U \subseteq \mathbb{R}^n$ is an open set that contains 0. Assume that $\mathrm{D}f(0) : \mathbb{R}^n \to \mathbb{R}^n$ is a linear diffeomorphism, then f has a local inverse f^{-1} at 0 and f^{-1} is of class $C^{k,\delta}$.

In the following subsections we will recall two versions of Peres and Schlag's projection theorem: one for projection families with a one-dimensional parameter space (Theorem 3.7) and one for projection families with a higher-dimensional parameter space (Theorem 3.11). As we will see, the one-dimensional case is contained in the higher dimensional case. However, since the notation in the setting of a one-dimensional parameter space is slimmer and the conditions seem more intuitive, we discuss this case separately first.

3.2.1 *One-dimensional parameter spaces*

Let (Ω, d) be a compact metric space and $J \subset \mathbb{R}$ an open interval. Then, we call a continuous mapping

$$\Pi : J \times \Omega \to \mathbb{R}, \quad (\lambda, \omega) \mapsto \Pi(\lambda, \omega), \tag{3.3}$$

a (one-parameter) family of abstract projections. We will actually think of Π as the family of mappings $\{\Pi_\lambda \Omega \to \mathbb{R} : \lambda \in J\}$ where $\Pi_\lambda(\omega) := \Pi(\lambda, \omega)$ for all $\lambda \in J$ and $\omega \in \Omega$. For $\lambda \in J$ and $\omega_1, \omega_2 \in \Omega$ two distinct points, we define

$$\Phi(\lambda, \omega_1, \omega_2) = \frac{\Pi(\lambda, \omega_1) - \Pi(\lambda, \omega_2)}{\mathrm{d}(\omega_1, \omega_2)}. \tag{3.4}$$

This makes Φ a mapping $J \times ((\Omega \times \Omega) \setminus \mathrm{Diag}) \to \mathbb{R}$ where Diag denotes the diagonal of the product space $\Omega \times \Omega$.

As we shall see in Theorem 3.7, the following definition represents a sufficient condition for certain Marstrand-type results to hold for a family Π of abstract projections.

Definition 3.5. We say that a family of abstract projections $\Pi : J \times \Omega \to \mathbb{R}$ satisfies differentiable transversality if there exists a positive integer L and some $0 \leq \delta \leq 1$, such that $L + \delta > 1$, Π is L-times continuously differentiable in the first variable, and the following hold:

(a) for any compact interval $I \subset J$,
  - for all $l = 1, 2, \ldots, L$, $\frac{\mathrm{d}^l}{\mathrm{d}\lambda^l}\Pi : I \times \Omega \to \mathbb{R}$ is bounded,
  - for all $\omega \in \Omega$, $\lambda \mapsto \frac{\mathrm{d}^L}{\mathrm{d}\lambda^L}\Pi(\lambda, \omega)$ is δ-Hölder on I with multiplicative constant independent of ω,

(b) the following transversality condition is satisfied: there exists a constant $C > 0$, such that for all pairs of distinct points $\omega_1, \omega_2 \in \Omega$ and $\lambda \in J$, for which $|\Phi(\lambda, \omega_1, \omega_2)| \leq C$,
$$\left| \frac{\mathrm{d}}{\mathrm{d}\lambda}\Phi(\lambda, \omega_1, \omega_2) \right| \geq C.$$

(c) there exist constants $\widetilde{C} > 0$ and $\widetilde{C}_l > 0$, for $l = 1, \ldots, L$, such that: if for some $\omega_1 \neq \omega_2 \in \Omega$ and $\lambda_1, \lambda_2 \in J$, we have $|\Phi(\lambda_1, \omega_1, \omega_2)| + |\Phi(\lambda_2, \omega_1, \omega_2)| \leq C$, then
  - $\left| \frac{\mathrm{d}^l}{\mathrm{d}\lambda^l}\Phi(\lambda_1, \omega_1, \omega_2) \right| \leq \widetilde{C}_l$, for all $l = 1, 2, .., L$.
  - $\left| \frac{\mathrm{d}^L}{\mathrm{d}\lambda^L}\Phi(\lambda_1, \omega_1, \omega_2) - \frac{\mathrm{d}^L}{\mathrm{d}\lambda^L}\Phi(\lambda_2, \omega_1, \omega_2) \right| \leq \widetilde{C} \left|\lambda_1 - \lambda_2\right|^\delta$.

Remark 3.6.

(i) The condition that $L + \delta > 1$ rules out the case where Π is differentiable in the first variable but its derivatives are not locally δ-Hölder for any $\delta > 0$.

(ii) The constant C appearing in (c) is the transversality constant defined in (b).

(iii) We allow the value ∞ for L. In this case the second conditions in (a) and (c) should be omitted.

(iv) In case $\delta = 0$, the second condition in (a) as well as the second condition in (c) is obsolete.

(v) For all $\widetilde{L} < L$ and $0 \leq \delta \leq 1$, differentiable transversality with constants L and 0 implies differentiable transversality with constants $\widetilde{L}$ and δ.

Theorem 3.7. *Let $\Pi : J \times \Omega \to \mathbb{R}$ be a family of abstract projections that satisfies differential transversality for constants $L \in \mathbb{N}$ and $0 \leq \delta \leq 1$ with $L + \delta > 0$. Moreover, assume that for all $\lambda \in J$, $\Pi_\lambda : \Omega \to \mathbb{R}$ is dimension non-increasing. Then, the following hold for all Borel sets $A \subseteq \Omega$.*

(1) *If $\dim A \leq 1$, then*

  (a) $\dim(\Pi_\lambda A) = \dim A$ *for $\mathscr{L}^1$-a.e. $\lambda \in J$,*

  (b) For $0 < \alpha \leq \dim A$, $\dim(\{\lambda \in J : \dim(\Pi_\lambda A) < \alpha\}) \leq \alpha$.

(2) *If $\dim A > 1$, then*

  (a) $\mathscr{L}^1(\Pi_\lambda A) > 0$ *for $\mathscr{L}^1$-a.e. $\lambda \in J$,*

  (b) $\dim(\{\lambda \in J : \mathscr{L}^1(\Pi_\lambda A) = 0\}) \leq 2 - \min\{\dim A, L + \delta\}$.

(3) *If $\dim A > 2$, then*

  (a) $\Pi_\lambda A \subset \mathbb{R}$ *has non-empty interior for $\mathscr{L}^1$-a.e. $\lambda \in J$,*

  (b) $\dim(\{\lambda \in J : (\Pi_\lambda A)^\circ \neq \varnothing\}) \leq 1 - (\min\{\dim A, L + \delta\} - 2)(1 + \frac{1}{L+\delta})^{-1}$.

Remark 3.8.

(i) If we dropped the assumption of $P_\lambda : \Omega \to \mathbb{R}$ being dimension non-increasing for $\lambda \in J$, Theorem 3.7 still holds with (1.a) changed to: $\dim(\Pi_\lambda A) \geq \dim A$ for $\mathscr{L}^1$-a.e. $\lambda \in J$,

(ii) We will mostly apply Theorem 3.7 in settings with high regularity:
In case that $L+\delta \geq 2$, (2.b) becomes: $\dim(\{\lambda \in J : \mathscr{L}^1(\Pi_\lambda A) = 0\}) \leq 2 - \dim A$.
And in case $L = \infty$, (3.b) becomes: $\dim(\{\lambda \in J : (\Pi_\lambda A)^\circ \neq 0\}) \leq 3 - \dim A$.

(iii) Theorem 3.7 is a special case of Theorem 4.9 in [34].

3.2.2 *Higher-dimensional parameter spaces*

In this section we are going to recall the higher-dimensional version of Definitions and Theorems from the previous section and state an additional consequence of differentiable transversality.

Recall that m and n are positive integers with $n > m$. Let K be another integer with

$$K \geq m \geq 1.$$

Let (Ω, d) be a compact metric space and $Q \subseteq \mathbb{R}^K$ an open connected set. We call a continuous mapping

$$\Pi : Q \times \Omega \to \mathbb{R}^m, \quad (\lambda, \omega) \mapsto \Pi(\lambda, \omega) \tag{3.5}$$

a (higher-dimensional) family of abstract projections. As in the one dimensional case, we indeed think of P as a family of mappings $\{\Pi_\lambda : \Omega \to \mathbb{R}^m : \lambda \in Q\}$ where $\Pi_\lambda(\omega) := \Pi(\lambda, \omega)$ for all $\omega \in \Omega$ and $\lambda \in Q$.

For $\omega_1 \neq \omega_2 \in \Omega$ and $\lambda \in Q$, define

$$\Phi(\lambda, \omega_1, \omega_2) := \frac{\Pi(\lambda, \omega_1) - \Pi(\lambda, \omega_2)}{d(\omega_1, \omega_2)} \in \mathbb{R}^m \tag{3.6}$$

Let us introduce the following notation for derivatives in higher-dimensional Euclidean space: For a function $f : \mathbb{R}^n \to \mathbb{R}^m$, we denote the differential of f in a point $x \in \mathbb{R}^n$ by $\mathrm{D}f(x)$. Moreover, we denote by $\frac{\partial}{\partial x_i} f(x)$ the (first order) partial derivative of f with respect to the i-th component in the point x. Since we will mostly consider continuously differentiable functions f, we will not distinguish between $\mathrm{D}f(x)$ and the Jacobian matrix of f in x. (i.e. the matrix whose entries are the first order partial derivatives of f). For higher-order partial derivatives we will use the following standard notation with multi-indices: For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, we write $|\alpha| := \alpha_1 + \ldots + \alpha_N$ and for $x \in \mathbb{R}^n$ and a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$, we define,

$$\frac{\partial^\alpha}{\partial x^\alpha} f(x) := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdot \ldots \cdot \partial x_n^{\alpha_n}} f(x).$$

Now, we can formulate the following analog of Definition 3.5:

Definition 3.9. We say that a family of abstract projections $\Pi : Q \times \Omega \to \mathbb{R}^m$ satisfies differentiable transversality if there exists a positive integer L and some $0 \leq \delta \leq 1$ such that $L + \delta > 1$, Π is L-times continuously differentiable in the first variable $\lambda \in Q$, and the following hold:

(a) for any compact connected subset $Q' \subset Q$,
   - for all α with for all $|\alpha| \leq L$: $\frac{\partial^\alpha}{\partial \lambda^\alpha} \Pi : Q' \times \Omega \to \mathbb{R}^m$ is bounded,
   - for all $\omega \in \Omega$ and $|\alpha| = L$, $\lambda \mapsto \frac{\partial^\alpha}{\partial \lambda^\alpha} \Pi(\lambda, \omega)$ is δ-Hölder on Q' with multiplicative constant independent of ω,

(b) the following transversality condition is satisfied: there exists a constant $C > 0$, such that for all pairs of distinct points $\omega_1, \omega_2 \in \Omega$ and $\lambda \in Q$ for which $|\Phi(\lambda, \omega_1, \omega_2)| \leq C$, it follows that

$$\left|\det \mathrm{D}_\lambda \Phi(\lambda,\omega_1,\omega_2)(\mathrm{D}_\lambda \Phi(\lambda,\omega_1,\omega_2))^{\mathsf{T}}\right| \geq C,$$

where $(\mathrm{D}_\lambda \Phi(\lambda,\omega_1,\omega_2))^{\mathsf{T}}$ denotes the transpose of the matrix $\mathrm{D}_\lambda \Phi(\lambda,\omega_1,\omega_2)$.

(c) there exist constant $\widetilde{C} > 0$ and $\widetilde{C}_l > 0$, for $l = 1,\ldots,L$, such that whenever $|\Phi(\lambda_1,\omega_1,\omega_2)| + |\Phi(\lambda_2,\omega_1,\omega_2)| \leq C$ for $\omega_1 \neq \omega_2 \in \Omega$ and $\lambda_1,\lambda_2 \in Q$, then:

- $\left|\frac{\partial^\alpha}{\partial\lambda^\alpha}\Phi(\lambda_1,\omega_1,\omega_2)\right| \leq \widetilde{C}_l$, for all $|\alpha| \leq L$,
- $\left|\frac{\partial^\alpha}{\partial\lambda^\alpha}\Phi(\lambda_1,\omega_1,\omega_2) - \frac{\partial^\alpha}{\partial\lambda^\alpha}\Phi(\lambda_2,\omega_1,\omega_2)\right| \leq \widetilde{C}\,|\lambda_1-\lambda_2|^\delta$, for all $|\alpha| = L$,

As in the one-dimensional setting in the previous section, we allow the value ∞ for L. Then, the second conditions in (a) and (c) may be neglected.

Remark 3.10. Note that in the special case when $m = n-1$, the matrix $\mathrm{D}_\lambda \Phi(\lambda,\omega_1,\omega_2)$ appearing in condition (b) of Definition 3.9, is an $(m \times m)$-matrix. Thus, by setting $C' := \min\{C, \sqrt{C}\}$, (b) is equivalent to:

(b') There exists a constant $C' > 0$, such that for all pairs of distinct points $\omega_1,\omega_2 \in \Omega$ and $\lambda \in Q$ for which $|\Phi(\lambda,\omega_1,\omega_2)| \leq C'$, it follows that

$$|\det \mathrm{D}_\lambda \Phi(\lambda,\omega_1,\omega_2)| \geq C'.$$

The following theorem is a generalization of Theorem 3.7 to higher dimensional parameter (and target) space:

Theorem 3.11. *Let $\Pi : Q \times \Omega \to \mathbb{R}^m$ be a family of abstract projections that satisfies differentiable transversality. Moreover, assume that for all $\lambda \in Q$, $\Pi_\lambda : \Omega \to \mathbb{R}^m$ is dimension non-increasing. Then, the following statements hold for all Borel sets $A \subseteq \Omega$.*

(1) *If $\dim A \leq m$, then*
   (a) $\dim(\Pi_\lambda A) = \dim A$ *for $\mathscr{L}^K$-a.e. $\lambda \in Q$,*
   (b) *For $0 < \alpha \leq \dim A$,* $\dim(\{\lambda \in Q \ : \ \dim(\Pi_\lambda A) < \alpha\}) \leq (n-m-1)m + \alpha.$

(2) *If $\dim A > m$, then*
   (a) $\mathscr{L}^m(\Pi_\lambda A) > 0$ *for $\mathscr{L}^K$-a.e. $\lambda \in Q$,*
   (b) $\dim(\{\lambda \in Q \ : \ \mathscr{L}^m(\Pi_\lambda A) = 0\}) \leq (n-m)m + m - \min\{\dim A, L+\delta\}.$

(3) *If $\dim A > 2m$, then*
   (a) $\Pi_\lambda A \subset \mathbb{R}^m$ *has non-empty interior for $\mathscr{L}^K$-a.e. $\lambda \in Q$,*
   (b) $\dim(\{\lambda \in Q : (\Pi_\lambda A)^\circ \neq \varnothing\})$
   $\leq (n-m)m - (\min\{\dim A, L+\delta\} - 2m)(1 + \frac{m}{L+\delta})^{-1}$

Remark 3.12.

(i) Choosing $m = K = 1$ in Theorem 3.11 yields Theorem 3.7.

(ii) Theorem 3.11 remains true if we drop the assumption that the projections $\Pi_\lambda : \Omega \to \mathbb{R}^m$ are dimension non-increasing. However, in this case, (1.a) becomes: $\dim(\Pi_\lambda A) \geq \dim A$ for $\mathscr{L}^m$-a.e. $\lambda \in Q$.

(iii) We will mostly apply Theorem 3.11 in settings with high regularity.
In case that $L + \delta \geq n$, (2.b) becomes:

$$\dim(\{\lambda \in Q \ : \ \mathscr{L}^m(\Pi_\lambda A) = 0\}) \leq (n-m)m + m - \dim A.$$

And in case $L = \infty$, (3.b) becomes:

$$\dim(\{\lambda \in Q : (\Pi_\lambda A)^\circ \neq 0\}) \leq (n-m)m + 2m - \dim A.$$

(iv) Definition 3.9 and Theorem 3.11 correspond to Definitions 7.1 and 7.2, and Theorem 7.3 in [34].

Remark 3.13. Recall from (3.1) that by $P : G(n,m) \times \mathbb{R}^n \to \mathbb{R}^n$ we denote the family of Euclidean projections $P_V : \mathbb{R}^n \to V$ onto m-planes $V \in G(n,m)$. Moreover, recall from Section 2.3, that $\varphi : \mathrm{Mat}_{(n-m)\times m}(\mathbb{R}) \to G(n,m)$ is a smooth local parameterization of $G(n,m)$ and that for all $T \in \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$, the vectors $e_1^T, \ldots, e_m^T$ form an orthonormal basis of $\varphi(T) \in G(n,m)$ that varies smoothly in T. Furthermore, $w_1, \ldots, w_m$ denotes the standard basis of $\mathbb{R}^m$.
Let $\Omega \subset \mathbb{R}^n$ be a compact set and consider the family of abstract Euclidean projections $\Pi^{\mathbb{E}} : Q \times \Omega \to \mathbb{R}^m$ defined by

$$\Pi^{\mathbb{E}}(T,x) := \sum_{i=1}^{m} \langle P^{\mathbb{E}}(\varphi(T),x), e_i^T \rangle w_i, \tag{3.7}$$

where $\langle \cdot, \cdot \rangle$ denote the Euclidean inner product (scalar product) in $\mathbb{R}^n$. Thus, the mapping $\Pi^{\mathbb{E}}$ is the mapping $P^{\mathbb{E}}$ restricted to $\varphi(\mathrm{Mat}_{(n-m)\times m}(\mathbb{R})) \times \Omega$ where V is identified with $\mathbb{R}^m$ is a smooth way. It can be shown by a straight-forward calculation that the family of abstract Euclidean projections satisfies differentiable transversality with $L = \infty$ and hence, all conclusions from Theorem 3.7 hold for $\Pi^{\mathbb{E}} : Q \times \Omega \to \mathbb{R}^m$ with $L = \infty$.

Furthermore, Hovila et. al. [20] have shown that in case of a slightly modified version of differentiable transversality, one obtains a Besicovitch-Federer characterization of purely unrectifiable sets; compare Theorem 3.4.

Theorem 3.14. *Assume that $\Pi : Q \times \Omega \to \mathbb{R}^m$ is both, a continuously differentiable map on $Q \times \Omega$ and a family of abstract projections that satisfies differentiable transversality with $L = 2$ (and $\delta = 0$). Then, each $\mathscr{H}^m$-measurable set $A \subseteq \mathbb{R}^n$ with $\mathscr{H}^m(A) < \infty$ is purely m-unrectifiable if and only if $\mathscr{H}^m(\Pi_V(A)) = 0$ for $\sigma_{n,m}$-a.e. $V \in G(n,m)$.*

The Euclidean version of this result is sometimes also referred to as the Besicovitch-Federer projection theorem, see Theorem 18.1 in [30].

In this chapter, we compare different methods of proof for Marstrand-type projection theorems. In particular, we will discuss the two notions of transversality introduced in Chapter 2. These are metric transversality (Definition 3.2) for a family of projections $P : G(n,m) \times \mathbb{R}^n \to \mathbb{R}^n$, $P(V,x) \in V$, and differentiable transversality (Definitions 3.5 and 3.9) for a family of abstract projections $\Pi : Q \times \Omega \to \mathbb{R}^m$ where $Q \subseteq \mathbb{R}^K$ open and Ω is a compact metric space; see (3.3) and (3.5).

Intuitively spoken, in order to obtain Marstrand-type projection theorems for a family of projections, one has to control the quantity of projections $P_V : \mathbb{R}^n \to V$, for which (many) pairs of distinct points get mapped to the same point or very close to each other. Both types of transversality provide such a control; while metric transversality literally bounds the size of the set of planes $V \in G(n,m)$ for which an arbitrary pair of distinct points gets mapped δ-close, differentiable transversality is concerned with the ratio of the distance of two projected points and the distance of the points themselves. Namely, it imposes that if this ratio is small, then it grows fast (for a sufficiently large number of directions) when the projection parameter is altered (in this direction). Thus, it is natural to examine how these notions of transversality are related.

Let us formally relate the notion of a family of projections $P : G(n,m) \times \mathbb{R}^n \to \mathbb{R}^n$ to the notion of a family of abstract projections $\Pi : Q \times \Omega \to \mathbb{R}^m$. We will do so by locally identifying $G(n,m)$ with $\mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$ which again is identified with $\mathbb{R}^K$ where $K = (n-m)m$ (see Section 2.3), and by identifying each m-plane V with $\mathbb{R}^m$ in a smooth way. To do so, let $P : G(n,m) \times \mathbb{R}^n \to \mathbb{R}^n$ be a family of projections such that $P(V,x) \in V$ for all $x \in \Omega$, $V \in G(n,m)$. Let $Q = \mathbb{R}^{(n-m)m} = \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$ and let

$$\varphi : \mathrm{Mat}_{(n-m)\times m}(\mathbb{R}) \to G(n,m),\ T \mapsto V_T$$

as defined in (2.4) be a local parameterization of $G(n,m)$. Moreover, by $e_1^T, \dots, e_m^T$ denote the orthonormal basis of V_T defined in (2.5) and recall that the vectors e_i^T vary smoothly in T. Recall that by $w_1, \dots, w_m$ we denote the standard (orthonormal) basis of $\mathbb{R}^m$. Let $\Omega \subset \mathbb{R}^n$ be a large ball centered at the origin and set $Q = \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$. Recall that $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product (scalar product) in $\mathbb{R}^n$. We define the family of abstract projections $\Pi^P : \mathrm{Mat}_{(n-m)\times m}(\mathbb{R}) \times \Omega \to \mathbb{R}^m$ by

$$\Pi^P(T,x) := \sum_{i=1}^{m} \langle P(V_T,x), e_i^T \rangle w_i. \tag{3.8}$$

for all $x \in \Omega$ and $\lambda \in \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$. In other words, $\Pi^P(T,x)$ is the a vector in $\mathbb{R}^m$ with entries $(\Pi(T,x))_i = \langle P(V_T,x), e_i^T \rangle$. Note that this makes Π^P a family of abstract projections in the sense of (3.6) and (3.3).

In case $n = 2$ and $m = 1$, the parameter space $\mathbb{R}^{(n-m)m} = \mathbb{R}$ is one-dimensional. Hence, in this case, every matrix $T \in \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$ is a number $t \in \mathbb{R}$. Moreover, a connected subset Q of the parameter space $\mathbb{R}$ is an interval and will be denoted by I.

The following lemma is a direct consequence of the fact that the mapping $V_T \to \mathbb{R}^m$ given by $u \mapsto \sum_{i=1}^{m} \langle u, e_i^T \rangle w_i$ is 1-bi-Lipschitz, for every $T \in \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$.

Lemma 3.15. *For all Borel sets $A \subseteq \Omega \subset \mathbb{R}^n$ and all parameters $T \in \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$,*

- $|\Pi^P(T,x) - \Pi^P(T,y)| = |P(V_T,x) - P(V_T,y)|$*, for all $x, y \in \Omega$,*
- $\mathscr{H}^s(\Pi_T^P(A)) = \mathscr{H}^s(P_{V_T}(A))$*, for all $s > 0$,*
- $\dim(\Pi_T^P(A)) = \dim(P_{V_T}(A))$.

The following proposition is an immediate consequence of Lemma 3.15 and Remark 2.1.

Proposition 3.16. *The conclusions of Theorem 3.11 hold for $\Pi^P : \mathrm{Mat}_{(n-m)\times m}(\mathbb{R}) \times \Omega \to \mathbb{R}^m$ ($Q = \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$) if and only if they hold for $P : \varphi(Q) \times \Omega \to \mathbb{R}^n$ where the term "for $\mathscr{L}^K$-a.e." is replaced by "$\sigma_{n,m}$-a.e." in the statements (a).*

The above proposition makes it plausible to compare the notions of transversality formulated for families of projections P and Π (resp. Π^P). As the following proposition shows, for projections families with a one-dimensional parameter space, differentiable transversality for Π^P implies metric transversality for P. This matches our observation that the conclusions of Theorem 3.3 (which follow from metric transversality) are weaker than the conclusions of Theorem 3.7 (which follow from differentiable transversality).

Proposition 3.17. *Consider a family of projections $P : G(2,1) \times \mathbb{R}^2 \to \mathbb{R}^2$ and the according family of abstract projections $\Pi^P : \mathbb{R} \times \Omega \to \mathbb{R}$. Suppose that $\Pi^P : \mathbb{R} \times \Omega \to \mathbb{R}$ is continuous and that it is C^1 in the first variable. Furthermore, assume that Π^P satisfies condition (b) as well as the first part of condition (c) from Definition 3.5. Then, for all compact subintervals $\widetilde{J} \subset J$, the restriction of P to $\varphi(\widetilde{J}) \times \Omega$ satisfies Definition 3.2.*

We conjecture that this is also true in higher dimensions, however our method of proof does not allow a generalization to higher dimensions. We will get back to this towards the end of this section.

Proof. Let $\widetilde{J} \subset J$ be a compact subinterval. First, notice that since Π^P is continuous, also P is continuous on $\varphi(\widetilde{J}) \times \Omega$. This suffices for condition (a) in Definition 3.2 to hold. Towards the proof of condition (b), note that by Lemma 3.15, we have $|\Pi^P(t,x) - \Pi^P(t,y)| = |P(V_t,x) - P(V_t,y)|$ for $t \in \mathbb{R}$. Therefore, it suffices to show that there exist constants $K > 0$ and $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$,

$$\mathscr{L}^1(\{t \in \widetilde{J} : |\Phi^P(t,x,y)| \leq \epsilon\}) \leq K\epsilon, \tag{3.9}$$

for all $x \neq y \in \Omega$, $t \in I$, and $\Phi^P(t,x,y) = \frac{\Pi^P(t,x) - \Pi^P(t,y)}{|x-y|}$ as in (3.4).

For the proof of (3.9), let $C > 0$ be as in Definition 3.5 and fix $x \neq y \in \Omega$. For $0 < \epsilon < C$, define $\mathcal{A}(\epsilon)$ to be the collection of open intervals $I \subset \widetilde{J}$ such that:

- for all $t \in I$: $|\Phi^P(t,x,y)| < \epsilon$

– for all $t \in \partial I$: either $|\Phi^P(t,x,y)| = \epsilon$ or $t \in \partial \widetilde{J}$.

In particular, this makes $\mathcal{A}(\epsilon)$ a family of disjoint open intervals that cover the set $\{t \in \widetilde{J} : |\Phi^P(t,x,y)| < \epsilon\}$. Now, for $0 < \epsilon' < \epsilon < C$, we consider the following statements.

(I) Each Interval $I' \in \mathcal{A}(\epsilon')$ is contained in some interval $I \in \mathcal{A}(\epsilon)$

(II) Transversality: Each $I \in \mathcal{A}(\epsilon)$ contains at most one $I' \in \mathcal{A}(\epsilon')$.

Statement (I) is obvious. We now prove that Statement (II) follows from differentiable transversality. Let $I' \in \mathcal{A}(\epsilon')$, $I \in \mathcal{A}(\epsilon)$ such that $I' \subset I$. Then, by definition of $\mathcal{A}(\epsilon)$, it follows that $|\Phi^P(t,x,y)| < \epsilon < C$ for all $t \in I$. Then, by condition (b) of Definition (3.5) for Π^P, it follows that $|\frac{\mathrm{d}}{\mathrm{d}t}\Phi^P(t,x,y)| \geq C$ for all $t \in I$. Assume without loss of generality that $\frac{\mathrm{d}}{\mathrm{d}t}\Phi^P(t,x,y) > 0$ for all $t \in I$ (the opposite case is analogous). Thus, $t \mapsto \Phi^P(t,x,y)$ is strictly increasing for $t \in I$. Hence, by definition of I' and I, I' sits in the left most place within I. Thus, there cannot exist two disjoint such intervals $I' \in \mathcal{A}(\epsilon')$ within I. This proves Statement (II).

Based on Statements (I) and (II), we will first give an upper bound for the length of intervals $I \in \mathcal{A}(\epsilon)$, for $0 < \epsilon < C$, see (3.10). Then, we will give an upper bound for the number of elements of $\mathcal{A}(\frac{c}{4})$, see (3.11). The conclusion (3.12) we will draw from these estimates, proves (3.9) and thereby the proposition follows.

Let $0 < \epsilon < C$, $I \in \mathcal{A}(\epsilon)$ and $t_0 < t_1 \in I$. Since Π^P satisfies condition (b) from Definition 3.5 and $x, y \in I \in \mathcal{A}(\epsilon)$ we have

$$(t_1 - t_0)C = \int_{t_0}^{t_1} C\mathrm{d}t \leq \int_{t_0}^{t_1} \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,x,y)\mathrm{d}t = \Phi(t_1,x,y) - \Phi(t_0,x,y) < 2\epsilon.$$

Therefore, we obtain the following upper bound on the length of intervals $I \in \mathcal{A}(\delta)$:

$$\mathrm{length}(I) \leq \frac{2\epsilon}{C}. \tag{3.10}$$

Next, let $I \in \mathcal{A}(C)$ such that there exists $I'' \in \mathcal{A}(\frac{C}{4})$ such that $I'' \subset I$. Then, by the above Statement (II) this interval I'' is unique and there exists a unique interval $I' \in \mathcal{A}(\frac{C}{2})$ such that $I'' \subset I' \subset I$. Let $t_1 \in I \backslash I'$, $t_0 \in I''$ and without loss of generality assume that $t_0 < t_1$ (the opposite case works analogously). Notice that by boundedness of $\Phi^P(t,x,y)$ on compact sets (that is, the first part of (c) from Definition 3.5 for Π^P), it follows that:

$$\frac{C}{4} = \frac{C}{2} - \frac{C}{4} \leq \Phi(t_1,x,y) - \Phi(t_0,x,y) = \int_{t_0}^{t_1} \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,x,y)\mathrm{d}t \leq \int_{t_0}^{t_1} C_1\mathrm{d}t = (t_1 - t_2)C_1,$$

where $C_1 > 0$ is the upper bound of $|\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,x,y)|$ on $\widetilde{J} \times \Omega \times \Omega$. Hence, we obtain that,

$$\mathrm{length}(I) > \frac{C}{4C_1}$$

for all $I \in \mathcal{A}(C)$ for which there exists $I'' \in \mathcal{A}(\frac{C}{4})$ with $I'' \subset I$. Choose $N \in \mathbb{N}$ to be greater or equal than $\frac{4C_1}{C}\text{length}(\widetilde{J})$. Thus, the number of intervals $I \in \mathcal{A}(C)$ for which there exists $I'' \in \mathcal{A}(\frac{C}{4})$ with $I'' \subset I$, is smaller or equal to N.

Define $\epsilon_0 = \frac{C}{4}$ and by $\sharp\mathcal{A}(\epsilon)$ denote the number of elements in $\mathcal{A}(\epsilon)$. Then, by the Statements (I) and (II) above,

$$\sharp\mathcal{A}(\epsilon) < N \tag{3.11}$$

for all $0 < \epsilon < \epsilon_0$.

Finally, for all $0 < \epsilon < \epsilon_0$:

$$\mathscr{L}^1(\{t \in \widetilde{J} : \Phi(t,x,y) < \epsilon\}) = \mathscr{L}^1\Big(\bigcup_{I \in \mathcal{A}(\epsilon)} I\Big) \leq \sum_{I \in \mathcal{A}(\epsilon)} \mathscr{L}^1(I) \leq N\frac{2\epsilon}{C} \tag{3.12}$$

where the last inequality follows from (3.10) and (3.11). □

Remark 3.18.

(i) As we shall see in Chapter 5, Corollary 5.14, the converse of Proposition 3.17 does not hold: There exists a metrically transversal familiy of projections $P : G(n,m) \times \mathbb{R}^n \to \mathbb{R}^m$ such that Π^P fails to satisfy differentiable transversality for all choices of compact sets $\Omega \subset \mathbb{R}^n$.

(ii) We could not adapt the above proof of Proposition 3.17 to higher dimensional parameter space for several reasons. Here are three of them: First, condition (b) in Theorem 3.11 does not lead to an estimate of the distance (between t_0 and t_1) as directly as condition (b) in Theorem 3.7 does. Second, when the analogs of the sets $I \subset \mathcal{A}(\epsilon)$ are higher dimensional, estimating their diameter says little about the measure of the set. Third, since the analogs of the sets $I \subset \mathcal{A}(\epsilon)$ are not necessarily convex, we cannot bound the number of sets $I' \in \mathcal{A}(\epsilon')$ for which $I' \subset I$, as in the above proof.

Transversality has proven to be an important tool for establishing Marstrand-type projection theorems in various types of spaces. However, there are settings where (differentiable) transversality does not hold or leads to relatively weak results. Namely, in Chapter 4, we will see examples of families of linear and surjective projections (see Definition 4.1) for which differentiable transversality fails but the conclusions of Theorem 3.11 can be proven to hold by a comparison argument; see Corollary 5.14. Moreover, in hyperbolic space differentiable transversality holds but comparison with Euclidean projections with less effort yields stronger projection theorems; see Chapter 6.

Chapter 4

LINEAR PROJECTIONS IN $\mathbb{R}^n$

In this chapter, we will extend Marstrand-type projection theorems and transversality properties that are known to hold for the family of Euclidean projections to families of linear and surjective projections.

Definition 4.1. We call $P : G(n,m) \times \mathbb{R}^n \to \mathbb{R}^n$ a family of linear projections, if for every $V \in G(n,m)$, the mapping $P_V : \mathbb{R}^n \to V$ is a linear map. If in addition $P_V : \mathbb{R}^n \to V$ is surjective for all $V \in G(n,m)$, then we call $P : G(n,m) \times \mathbb{R}^n \to \mathbb{R}^n$ a family of linear and surjective projections in $\mathbb{R}^n$.

First notice that all linear maps are Lipschitz and therefore, every linear projection $P_V : \mathbb{R}^2 \to V$, $V \in G(n,m)$, is dimension non-increasing. Moreover, note that the family of Euclidean projections $P^{\mathbb{E}} : G(n,m) \times \mathbb{R}^n \to \mathbb{R}^n$ is an example of a family of linear and surjective projections.

In the first part of the chapter, we will give an (essentially sharp) condition that guarantees Marstrand-type projection theorems for families of linear projections. Then, in the second part, we will give a list of properties that guarantee that a given family of linear projections satisfies differentiable transversality. Chapter 5 will provide many concrete examples of families of linear (and surjective) projections for which the results of this chapter apply. Moreover, in Chapter 5.4, we will construct a family of linear and surjective projections for which Marstrand's theorem fails.

4.1 PROJECTION THEOREMS VIA COMPARISON

In order to establish strong Marstrand-type results for families $P : G(n,m) \times \mathbb{R}^n \to \mathbb{R}^n$ of linear and surjective projections, it turns out to be useful to compare them to the family of Euclidean projections: for all $V \in G(n,m)$ we will choose a particular $V' \in G(n,m)$ so that P_V is comparable to the Euclidean projection $P^{\mathbb{E}}_{V'}$ in terms of measure and dimension of projected sets. Then, in order to guarantee that the desired projection theorems hold for P we need to ensure that the mapping that associates V' to V has good measure theoretic properties. We start by formally defining the mapping $\mathscr{G}$ that associates an m-plane V' to each m-plane V.

Let $P : G(n,m) \times \mathbb{R}^n \to \mathbb{R}^n$ be a family of linear and surjective projections and let $V \in G(n,m)$. Then, $P_V^{-1}(\{0\}) = \mathrm{Kern} P_V$ is an element of $G(n, n-m)$.

Define

$$\mathscr{G}(V) := (P_V^{-1}(\{0\}))^{\perp} = (\operatorname{Ker} P_V)^{\perp}. \tag{4.1}$$

This makes $\mathscr{G}(V)$ an element of $G(n,m)$ and we can view $\mathscr{G}$ as a mapping $\mathscr{G} : G(n,m) \to G(n,m)$; see Figure 4.1.

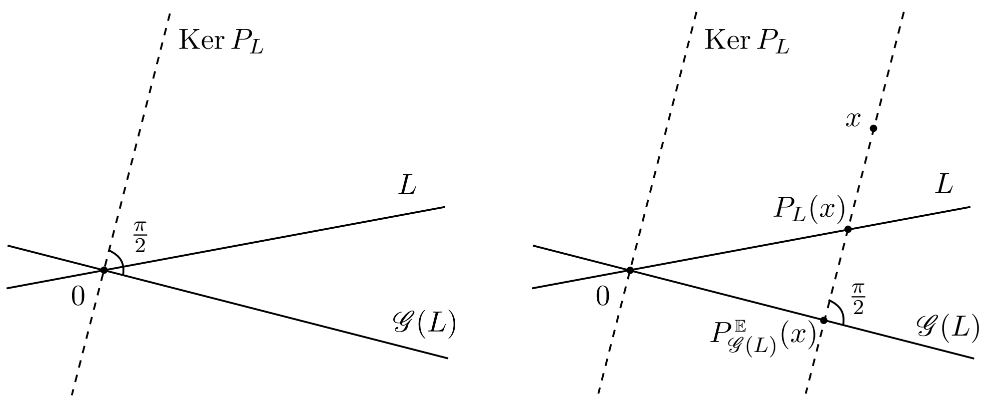

Figure 4.1. The mapping $\mathscr{G}$ on $G(2,1)$ and the linear projections P and $P^{\mathbb{E}}$.

The comparison of P_V and $P^{\mathbb{E}}_{\mathscr{G}(V)}$ will lead to the following theorem.

Theorem 4.2. *Assume that $P : G(n,m) \times \mathbb{R}^n \to \mathbb{R}^n$ is a family of linear and surjective projections and that the associated mapping $\mathscr{G} : G(n,m) \to G(n,m)$ is dimension non-decreasing and has the inverse Lusin property for the measure $\sigma_{n,m}$. Then, the following hold for all Borel sets $A \subseteq \mathbb{R}^n$.*

(1) *If* $\dim A \le m$, *then*

  (a) $\dim(P_V A) = \dim A$ *for* $\sigma_{n,m}$*-a.e.* $V \in G(n,m)$,

  (b) *For* $0 < \alpha \le \dim A$,
  $\dim(\{V \in G(n,m) : \dim(P_V A) < \alpha\}) \le (n-m-1)m + \alpha$.

(2) *If* $\dim A > m$, *then*

  (a) $\mathscr{H}^m(P_V A) > 0$ *for* $\sigma_{n,m}$*-a.e.* $V \in G(n,m)$,

  (b) $\dim(\{V \in G(n,m) : \mathscr{H}^m(P_V A) = 0\}) \le (n-m)m + m - \dim A$.

(3) *If* $\dim A > 2m$, *then*

  (a) $P_V A \subseteq V \simeq \mathbb{R}^m$ *has non-empty interior for* $\sigma_{n,m}$*-a.e.* $V \in G(n,m)$,

  (b) $\dim(\{V \in G(n,m) : (P_V A)^{\circ} \neq \varnothing\}) \le (n-m)m + 2m - \dim A$.

The definition of the inverse Lusin property was given in Section 2.1. In order to prove Theorem 4.2, we employ the following lemma:

Lemma 4.3. *Let $f : \mathbb{R}^n \to \mathbb{R}^d$ and $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}^m$ be linear mappings with* $\operatorname{Ker} f = \operatorname{Ker} \widetilde{f}$. *Then, there exists a bijective linear mapping $h : f(\mathbb{R}^n) \to \widetilde{f}(\mathbb{R}^n)$ such that for all $x \in \mathbb{R}^n$,*

$h(f(x)) = \widetilde{f}(x)$. *Thus, in particular, for every* $A \subseteq \mathbb{R}^n$, $h(f(A)) = \widetilde{f}(A)$ *and hence* $f(A)$ *and* $\widetilde{f}(A)$ *have the same Hausdorff measure and dimension.*

Proof. In case $V := \operatorname{Ker} f = \operatorname{Ker} \widetilde{f}$ equals $\mathbb{R}^n$ or $\{0\}$, the Lemma is trivial. Therefore, we may assume without loss of generality that $0 < k := \dim(V) < n$. Let $v_1, \ldots, v_k$ be a basis of V and extend it to a basis $v_1, \ldots, v_k, w_1, \ldots, w_{n-k}$ of $\mathbb{R}^n$. Then, $f(w_1), \ldots, f(w_{n-k})$ is a basis of $f(\mathbb{R}^n)$ and $\widetilde{f}(w_1), \ldots, \widetilde{f}(w_{n-k})$ is a basis of $\widetilde{f}(\mathbb{R}^n)$. Define $h : f(\mathbb{R}^n) \to \widetilde{f}(\mathbb{R}^n)$ as follows: for $y \in f(\mathbb{R}^n)$, there is a unique choice of coefficients y_j, $j = 1, \ldots, n-k$, such that $y = \sum_{j=1}^{n-k} y_j f(w_j)$. Set

$$h(y) := \sum_{j=1}^{n-k} y_j \widetilde{f}(w_j).$$

Then, h is a linear bijection and for every $x \in \mathbb{R}^n$, $x = \sum_{j=1}^{k} x_i v_i + \sum_{j=1}^{n-k} x_{k+j} w_j$ we have

$$h(f(x)) = h\Bigg(\sum_{j=1}^{n-k} x_{k+j} f(w_j)\Bigg) = \sum_{j=1}^{n-k} x_{k+j} \widetilde{f}(w_j) = \widetilde{f}(x).$$

□

Proof of Theorem 4.2. Let $A \subseteq \mathbb{R}^n$ be a Borel set and $0 < \alpha \leq \dim(A) \leq m$. We know that (1.a) and (1.b) of Theorem 4.2 hold for Euclidean projections, that is,

$$\sigma(\{W \in G(n,m) : \dim P_W^{\mathbb{E}}(A) < \alpha\}) = 0 \tag{4.2}$$

$$\dim(\{W \in G(n,m) : \dim P_W^{\mathbb{E}}(A) < \alpha\}) \leq \alpha. \tag{4.3}$$

By applying Lemma 4.3 for $f = P_V$ and $\widetilde{f} = P_{\mathscr{G}(V)}^{\mathbb{E}}$, it follows that, for all $V \in G(n,m)$,

$$\dim P_V(A) = \dim P_{\mathscr{G}(V)}^{\mathbb{E}}(A). \tag{4.4}$$

Notice that (4.2) yields

$$\begin{aligned}
&\sigma(\mathscr{G}\{V \in G(n,m) : \dim P_{\mathscr{G}(V)}^{\mathbb{E}}(A) < \alpha\}) \\
&= \sigma(\{\mathscr{G}(V) \in G(n,m) : \dim P_{\mathscr{G}(V)}^{\mathbb{E}}(A) < \alpha\}) \\
&\leq \sigma(\{W \in G(n,m) : \dim P_W^{\mathbb{E}}(A) < \alpha\}) \\
&= 0
\end{aligned}$$

Moreover, by (4.4), we know that

$$\sigma(\{V \in G(n,m) : \dim P_V(A) < \alpha\}) = \sigma(\{V \in G(n,m) : \dim P_{\mathscr{G}(V)}^{\mathbb{E}}(A) < \alpha\}).$$

Hence, by the fact that $\mathscr{G}$ has the inverse Lusin property, it follows that

$$\sigma(\{V \in G(n,m) : \dim P_V(A) < \alpha\}) = 0.$$

This proves (1.a). Furthermore, combining (4.3) and (4.4) with the fact that $\mathscr{G}$ is dimension non-decreasing, yields

$$\begin{aligned}
\dim(\{V \in G(n,m) : \dim P_V(A) < \alpha\}) &= \dim(\{V \in G(n,m) : \dim P^{\mathbb{E}}_{\mathscr{G}(V)}(A) < \alpha\}) \\
&\leq \dim(\mathscr{G}\{V \in G(n,m) : \dim P^{\mathbb{E}}_{\mathscr{G}(V)}(A) < \alpha\}) \\
&= \dim(\{\mathscr{G}(V) \in G(n,m) : \dim P^{\mathbb{E}}_{\mathscr{G}(V)}(A) < \alpha\}) \\
&\leq \dim(\{W \in G(n,m) : \dim P^{\mathbb{E}}_{W}(A) < \alpha\}) \\
&\leq \alpha.
\end{aligned}$$

This proves (1.b). The proofs of (2) and (3) are analogous. $\square$

Notice that in the proof of Theorem 4.2, we use the assumption that the mapping $\mathscr{G} : G(n,m) \to G(n,m)$ has the inverse Lusin property only for the parts (a). The assumption that $\mathscr{G}$ is dimension non-decreasing is used for the parts (b). Moreover, in order for Theorem 4.2 to hold, it suffices to assume these properties (dimension non-decreasingness and/or inverse Lusin property) for sets $E \subset G(n,m)$ that occur as exceptional sets of the family of Euclidean projections.
It is a trivial consequence of the proof of Theorem 4.2 that there exist many families of linear and surjective projections, for which all Marstrand-type theorems fail. For example, whenever $\mathscr{G}$ is constant in an open set of $G(n,m)$, all conclusions of Theorem 4.2 fail immediately. More generally, we can define a family of linear and surjective projections $P : G(n,m) \times \mathbb{R}^n \to \mathbb{R}^n$, by choosing a mapping $g : G(n,m) \to G(n,m)$ and setting $P_V(x) = P^{\mathbb{E}}_{\mathscr{G}(V)}(x)$.

Remark 4.4. Let $m = n-1$ and consider a family $P : G(n, n-1) \times \mathbb{R}^n \to \mathbb{R}^n$ of linear and surjective projections. Let $\widetilde{\mathscr{G}} : S^{n-1} \to S^{n-1}$ be any injective mapping that satisfies

$$\widetilde{\mathscr{G}}(v) \in \operatorname{Ker} P_{v^\perp} \tag{4.5}$$

Clearly such a mapping exists. Then, the mapping $\widetilde{\mathscr{G}}$ can be viewed as the mapping $\mathscr{G} : G(n, n-1) \to G(n, n-1)$, as defined in (4.1), under the identification of $G(n, n-1)$ with S^{n-1}; see Section 2.3. Thus, conclusions (1) and (2) from Theorem 4.2 hold for families of projections $P : G(n, n-1) \times \mathbb{R}^n \to \mathbb{R}^n$ for which $\widetilde{\mathscr{G}} : S^{n-1} \to S^{n-1}$ is dimension non-decreasing. Notice that conclusion (3) does not make sense for $m = n-1$.

Note that from the proof of Theorem 4.2 one can derive that Definition 3.2 (metric transversality) is satisfied for families of linear and surjective projections whose associated mapping $\mathscr{G}$ (resp. $\widetilde{\mathscr{G}}$) is dimension non-decreasing. However, the projection theorems one obtains from Theorem 3.1 by establishing metric transversality are weaker than the conclusions of Theorem 4.2. In particular, it is not known whether a statement like (2.b) in Theorem 4.2 can be derived from a notion such as metric transversality. Moreover, every known proof of (3), involves Fourier analytic methods and hence differentiability is required. Furthermore, we will see in the following section, in the case of linear projections, differentiable transversality only implies a weaker version of Theorem 4.2.

By the methods introduced in the proof of Theorem 4.2, we can deduce the following version of the Besicovitch-Federer projection theorem.

Theorem 4.5. *Assume that $P : G(n,m) \times \mathbb{R}^n \to \mathbb{R}^n$ is a family of linear and surjective projections whose associated mapping $\mathscr{G} : G(n,m) \to G(n,m)$ has the Lusin property as well as the inverse Lusin property, and that $\sigma_{n,m}(G(n,m) \backslash \mathscr{G}(G(n,m)) = 0$.*
Then, for all sets $A \subseteq \mathbb{R}^n$ with $\mathscr{H}^m(A) < \infty$, A is purely m-unrectifiable if and only if $\mathscr{H}^m(P_V(A)) = 0$ for $\sigma_{n,m}$-a.e. $V \in G(n,m)$.

Proof. Let $A \subseteq \mathbb{R}^n$ and define $E, E' \subset G(n,m)$ by

$$E := \{V \in G(n,m) : \mathscr{H}^m(P_V^{\mathbb{E}}(A)) = 0\}$$
$$E' := \{V \in G(n,m) : \mathscr{H}^m(P_V(A)) = 0\}$$

As in the proof of Theorem 4.2, by Lemma 4.3, it follows that $\mathscr{H}^m(P_V(A)) = 0$ if and only if $\mathscr{H}^m(P^{\mathbb{E}}_{\mathscr{G}(V)}(A)) = 0$. This yield that $E' = \{V \in G(n,m) : \mathscr{H}^m(P^{\mathbb{E}}_{\mathscr{G}(V)}(A)) = 0\}$ and thus $\mathscr{G}(E') \subseteq E$.
Now, assume that A is purely m-unrectifiable. Then, by Theorem 3.4, $\sigma_{n,m}(E) = 0$. Then, the fact that $\mathscr{G}$ has the inverse Lusin property implies that $\sigma_{n,m}(E') = 0$.
For the converse, assume that A is a set for which $\sigma_{n,m}(E) = 0$. From the assumption that $\sigma_{n,m}\big(G(n,m) \backslash \mathscr{G}(G(n,m))\big) = 0$ and the fact that $\mathscr{G}(E') \subseteq E$, it follows that $\sigma_{n,m}(\mathscr{G}(E')) = \sigma_{n,m}(E)$. And thus, since $\mathscr{G}$ has the Lusin property, we conclude that $\sigma(E) = 0$. ☐

4.2 DIFFERENTIABLE TRANSVERSALITY FOR LINEAR PROJECTIONS

Verifying differentiable transversality (Definition 3.5 resp. 3.9) for a given family of abstract projections is in general a non-trivial matter. It requires a lot of information of the nature of the projections in question (often one cannot get around finding an explicit formula for the projection) and involves verifying a number of technical conditions. However, in case the projection family is linear, the conditions from Definition 3.5 and 3.9 can be simplified by a considerable amount.

Let $\Omega \subset \mathbb{R}^n$ be a closed ball in $\mathbb{R}^n$ with radius $R > 1$ and center at $0 \in \mathbb{R}^n$. Moreover, let $Q \subseteq \mathbb{R}^K$ open and connected, and

$$\Pi : Q \times \Omega \to \mathbb{R}^m, \quad (\lambda, x) \mapsto \Pi(\lambda, x). \tag{4.6}$$

a family of projections as defined in (3.5).

Definition 4.6. We call $\Pi : Q \times \Omega \to \mathbb{R}^m$ a family of linear projections, if for every $\lambda \in Q$, the mapping $\Pi(\lambda, \cdot) : \Omega \to \mathbb{R}^m$ is the restriction of a linear mapping.

Proposition 4.7. *Let $\Pi : Q \times \Omega \to \mathbb{R}^m$ be a family of linear projections that satisfies the following properties:*

(P1) $\Pi : Q \times \Omega \to \mathbb{R}^m$ *is continuously differentiable in the first variable.*

(P2) For any compact connected subset $Q' \subset Q$,
- *for all α with for all $|\alpha| \le L$: $\frac{\partial^\alpha}{\partial \lambda^\alpha}\Pi : Q' \times \Omega \to \mathbb{R}^m$ is bounded,*
- *for all $x \in \Omega$ and $|\alpha| = L$, $\lambda \mapsto \frac{\partial^\alpha}{\partial \lambda^\alpha}\Pi(\lambda, x)$ is δ-Hölder on Q' with multiplicative constant independent of x.*

(P3) Whenever $\Pi(\lambda, x) = 0$ for some $x \in \Omega$ and $\lambda \in Q$, then

$$\left|\det\left(\mathrm{D}_\lambda \Pi(\lambda, x)(\mathrm{D}_\lambda \Pi(\lambda, x))^\mathsf{T}\right)\right| \ne 0.$$

Let $R \subset \mathbb{R}^K$ be an open and connected set such that $R \subset Q$ is compactly contained. Then, the family of projections $\Pi : R \times \Omega \to \mathbb{R}^m$ satisfies differentiable transversality for $L = 1$ and $\delta > 0$ (see Definition 3.9).

Proof. By Definition 4.6, we may assume that for every $\lambda \in Q$, Π_λ is a linear mapping $\mathbb{R}^n \to \mathbb{R}^m$ and thus C^∞. In particular, Π can be considered a mapping $Q \times \mathbb{R}^n \to \mathbb{R}^m$ that is linear in the second variable. Moreover, by property (P1), it is C^1 in the first variable. Thus, $\Pi : Q \times \mathbb{R}^n \to \mathbb{R}^m$ is a C^1-mapping and in particular, the mapping $\mathrm{D}_\lambda P : Q \times \mathbb{R}^n \to \mathbb{R}^{m \times m(n-m)}$, $(\lambda, x) \mapsto \mathrm{D}_\lambda P(\lambda, x)$ is continuous.
Obviously, property (P2) in Proposition 4.7 implies condition (a) of Definition 3.9 for $\Pi : R \times \Omega \to \mathbb{R}^m$. By linearity of $x \mapsto \Pi(\lambda, x)$, it follows that, for all $\lambda \in Q$ and $x_1, x_2 \in \mathbb{R}^n$ with $x_1 \ne x_2$,

$$\Phi(\lambda, x_1, x_2) = \frac{\Pi(\lambda, x_1) - \Pi(\lambda, x_2)}{|x_1 - x_2|} = \Pi\left(\lambda, \frac{x_1 - x_2}{|x_1 - x_2|}\right) \tag{4.7}$$

(see (3.4) for the definition of Φ). Hence, condition (c) of Definition 3.9 follows from property (P2) as well.
Note that by (4.7) and the fact that $\frac{x_1 - x_2}{|x_1 - x_2|} \in S^{n-1}$, in order to prove condition (b) from Definition 3.9 for the family $\Pi : R \times \Omega \to \mathbb{R}^m$, it suffices to show that: There exists a constant $C > 0$, such that whenever $|\Pi(\lambda, x)| \le C$ for some $x \in S^{n-1}$ and $\lambda \in \bar{R}$, then $\left|\det\left(\mathrm{D}_\lambda \Pi(\lambda, x)(\mathrm{D}_\lambda \Pi(\lambda, x))^\mathsf{T}\right)\right| \ge C$. Assume for a contradiction that this is false. Then, for every $n \in \mathbb{N}$, there exists a parameter $\lambda_n \in \bar{R}$ and a point $x_n \in S^{n-1}$ such that $|\Pi(\lambda_n, x_n)| \le \frac{1}{n}$ and $\left|\det\left((\mathrm{D}_\lambda \Pi(\lambda_n, x_n)(\mathrm{D}_\lambda \Pi(\lambda_n, x_n))^\mathsf{T})\right)\right| \le \frac{1}{n}$. Since $\bar{R} \times \Omega$ is compact, the sequence $(\lambda_n, x_n)_{n \in \mathbb{N}}$ admits a convergent subsequence with limit $(\lambda_0, x_0) \in \bar{R} \times S^{n-1}$. Then, by continuity of Π and $\mathrm{D}_\lambda \Pi$, it follows that $|\Pi(\lambda_0, x_0)| = 0$ and $\left|\det\left(\mathrm{D}_\lambda \Pi(\lambda_0, x_0)(\mathrm{D}_\lambda \Pi(\lambda_0, x_0))^\mathsf{T}\right)\right| = 0$ which contradicts (P3). □

Chapter 5

PROJECTIONS INDUCED BY A NORM

5.1 STRICTLY CONVEX NORMS AND PROJECTIONS

This section starts with a short introduction to convexity, the Gauss map of hypersurfaces, and norms in $\mathbb{R}^n$. Moreover, we will define families of closest-point projections with respect to strictly convex norms and establish some of their basic properties.

Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. We will denote spheres and (closed) balls with respect to $\|\cdot\|$ as follows.

$$\begin{aligned} S^{n-1}_{\|\cdot\|}(x,r) &= \{y \in \mathbb{R}^n : \|x-y\| = r\}, \\ B^n_{\|\cdot\|}(x,r) &= \{y \in \mathbb{R}^n : \|x-y\| \leq r\}. \end{aligned} \tag{5.1}$$

We will often abbreviate $S^{n-1}_{\|\cdot\|} = S^{n-1}_{\|\cdot\|}(0,1)$ and $B^n_{\|\cdot\|} = B^n_{\|\cdot\|}(0,1)$. Furthermore, we will denote the distance of two sets $A, B \subseteq \mathbb{R}^n$ with respect to $\|\cdot\|$ by

$$\operatorname{dist}_{\|\cdot\|}(A,B) = \inf\{\|a-b\| : a \in A, b \in B\}.$$

Recall that we use the symbol $|\cdot|$ for the Euclidean norm on $\mathbb{R}^n$. We will write S^{n-1} for the Euclidean unit sphere $S^{n-1}_{|\cdot|}(0,1)$, and B^n for the closed Euclidean unit ball $B^n_{|\cdot|}(0,1)$. We wish to recall the well-known fact that any two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbb{R}^n$ are bi-Lipschitz equivalent, that is, there exists a constant $L > 0$ such that for all $x, y \in \mathbb{R}^n$,

$$\frac{1}{L}\|x-y\|_1 \leq \|x-y\|_2 \leq L\|x-y\|_1.$$

This could be equivalently formulated as either of the following statements:

(a) The identity map $\mathbb{R}^n \to \mathbb{R}^n$ is a bi-Lipschitz map $(\mathbb{R}^n, \|\cdot\|_1) \to (\mathbb{R}^n, \|\cdot\|_2)$.

(b) The map $x \mapsto \frac{x}{\|x\|_1}$ is a bi-Lipschitz map from $(S^{n-1}, |\cdot|)$ onto $(S^{n-1}_{\|\cdot\|_1}, |\cdot|)$.

Recall that a closed set $F \subseteq \mathbb{R}^n$ is called convex (resp. strictly convex), if for all $x, y \in F$ and $t \in [0,1]$, the point $(1-t)x + ty$ is contained in F (resp. the interior of F). For a convex set $U \subseteq \mathbb{R}^n$, a function $f : U \to \mathbb{R}$ is called convex if for all points $x, y \in U$ and parameters $t \in [0,1]$,

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y).$$

The function f is called strictly convex, in case the inequality above is strict.

Since every norm is a convex and continuous function $\mathbb{R}^n \to [0,\infty)$, $B^n_{\|\cdot\|}$ is a compact and convex set with non-empty interior. Moreover, since every norm is symmetric (i.e. $\|x\| = \|-x\|$ for $x \in \mathbb{R}^n$), the ball $B^n_{\|\cdot\|}$ is antipodally symmetric (i.e., if $v \in \mathbb{R}^n$ is contained in $B^n_{\|\cdot\|}$, then so is $-v$). If in addition, $\|\cdot\|$ is strictly convex (i.e., the function $\|\cdot\| : \mathbb{R}^n \to [0,\infty)$ is strictly convex), then $B^n_{\|\cdot\|}$ is a strictly convex set. Conversely, it is known that every compact, (strictly) convex and antipodally symmetric set $B \in \mathbb{R}^n$ with non-empty interior, defines a (strictly convex) norm $\|\cdot\|_B$ on $\mathbb{R}^n$, by setting $\|x\|_B = |t|$ where $t \in \mathbb{R}$ with $tx \in \partial B$.

The following proposition is a simple consequence of the definitions above.

Proposition 5.1. *Let $\|\cdot\|$ be a strictly convex norm on $\mathbb{R}^n$ and let $A \subseteq \mathbb{R}^n$ be a closed and convex set. Then, there exists a unique closest point $q \in A$ to x, that is, there exists a unique $q \in A$ such that $\|x - q\| = \operatorname{dist}_{\|\cdot\|}(x, A)$.*

Consider a strictly convex norm $\|\cdot\|$ on $\mathbb{R}^n$ and note that m-planes in $\mathbb{R}^n$ are convex sets. Thus, for every $x \in \mathbb{R}^n$ and $V \in G(n,m)$, there exists a unique $q \in V$ that realizes the distance between x and V, that is, $\|q - x\| = \operatorname{dist}_{\|\cdot\|}(x, V)$. We denote this point q by $P_V^{\|\cdot\|}(x)$ and we define the family of closest-point projections for $\|\cdot\|$,

$$P^{\|\cdot\|} : G(n,m) \times \mathbb{R}^n \to \mathbb{R}^n \tag{5.2}$$

by $P^{\|\cdot\|}(V,x) = P_V^{\|\cdot\|}x$, for all $V \in G(n,m)$ and $x \in \mathbb{R}^n$. Notice that thus $P_V(x)$ is the unique point in the intersection $B^n_{\|\cdot\|}(x, \operatorname{dist}(x,V)) \cap V$, or equivalently, in the intersection $S^{n-1}_{\|\cdot\|}(x, \operatorname{dist}(x,V)) \cap V$; see left-hand side of Figure 5.1.

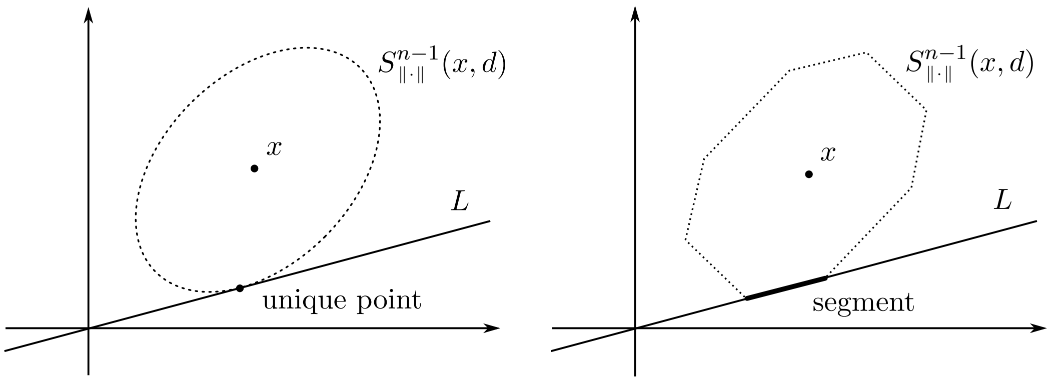

Figure 5.1. The set of closest points on L from x given as the intersection of the sphere $S^{n-1}_{\|\cdot\|}(x,d)$ with L ($d = \operatorname{dist}_{\|\cdot\|}(x,L)$) for two different norms.

We will often call $P^{\|\cdot\|}$ the family of projections induced by $\|\cdot\|$. Note that the family of projections $P^{|\cdot|}$ induced by the Euclidean norm $|\cdot|$ equals the family of Euclidean orthogonal projections $P^{\mathbb{E}}$.

If a norm $\|\cdot\|$ fails to be strictly convex, then the family of projections induced by $\|\cdot\|$ is not well-defined for some planes $V \in G(n,m)$. To see this, first notice that the boundary of every closed convex but not strictly convex set contains a line segment. Thus, if $\|\cdot\|$ is not strictly convex, $S^{n-1}_{\|\cdot\|}$ contains a line segment. Let $V \in G(n,m)$ contain the direction of this line segment. Then, for all $x \in \mathbb{R}^n$, there does not exist a unique point $q \in V$ that realizes the distance between V and x; see right-hand side of Figure 5.1 for an example.

One can equivalently define (strict) convexity of closed sets in $\mathbb{R}^n$ in terms of supportive hyperplanes. Namely, a closed set $F \subseteq \mathbb{R}^n$ with non-empty interior is convex if and only if every point in its boundary admits a supportive hyperplane, i.e., for all $x \in \partial F$, there exists an affine $(n-1)$-plane $H \subset \mathbb{R}^n$ that contains x so that F is contained in the closed half-space on one side of H. Moreover, F is strictly convex if in addition $F \cap H = \{x\}$. Notice that if the boundary ∂F of a non-empty, closed and convex set $F \subseteq \mathbb{R}^n$ with non-empty interior is an embedded $(n-1)$-dimensional differentiable manifold, then for each x the unique supportive hyperplane of F at x is $H = x + T_x\partial F$ where $T_x\partial F$ denotes the tangent plane of ∂F at x. Whenever the boundary of a non-empty open set $F \subseteq \mathbb{R}^n$ admits a tangent plane $T_x\partial F$ at a point x, the unit outward normal of ∂F at x is well-defined, i.e., there exists a unique $v \in S^{n-1}$ orthogonal to $T_x\partial F$ such that for all $r > 0$, we have $rv \notin F$. The mapping $G : \partial F \to S^{n-1}$ that maps $x \in \partial F$ to the unit outward normal v of ∂F at x, is called the Gauss map of ∂F.

Now, we apply these concepts to norms and their unit balls and spheres. Let $\|\cdot\|$ be a $C^{k,\delta}$-norm on $\mathbb{R}^n$, i.e., the restriction $\|\cdot\| : \mathbb{R}^n\backslash\{0\} \to [0,\infty)$ is of class $C^{k,\delta}$ for some $k \in \mathbb{N}$ and $\delta \geq 0$. Note that $S^{n-1}_{\|\cdot\|}$ is the preimage of the value 1 under the mapping $\|\cdot\| : \mathbb{R}^n \to [0,\infty)$. Therefore, $S^{n-1}_{\|\cdot\|}$ is an $(n-1)$-dimensional compact C^k-manifold in $\mathbb{R}^n$ and hence, the Gauss map G of $S^{n-1}_{\|\cdot\|}$ is a continuous mapping given by

$$G(x) = \frac{\nabla\|x\|}{|\nabla\|x\||} \tag{5.3}$$

where $\nabla\|x\|$ denotes the gradient of the mapping $\|\cdot\| : \mathbb{R}^n \to [0,\infty)$ at $x \in \mathbb{R}^n\backslash\{0\}$. We will often refer to $G : S^n_{\|\cdot\|} \to S^n$ as the Gauss map of $\|\cdot\|$. Recall that $S^n_{\|\cdot\|}$ is the boundary of the set $B^n_{\|\cdot\|}$ and that $B^n_{\|\cdot\|}$ is closed and convex, and has non-empty interior.

The following lemma lists some useful properties of the Gauss map G.

Lemma 5.2. *Let $\|\cdot\|$ be C^1-norm on $\mathbb{R}^n$. Then,*

(i) $\langle v, G(v)\rangle > 0$ *for all* $v \in S^{n-1}_{\|\cdot\|}$*,*

(ii) there exist two vectors $v_1, v_2 \in S^{n-1}_{\|\cdot\|}$*,* $v_1 \neq \pm v_2$*, such that* $G(v_1)$ *is collinear with* v_1 *and* $G(v_2)$ *is collinear with* v_2
(and hence, by symmetry, $G(-v_i)$ *is collinear with* v_i*, for* $i = 1, 2$*),*

(iii) $G : S^{n-1}_{\|\cdot\|} \to S^{n-1}$ *is surjective,*

(iv) G *is injective if and only if* $\|\cdot\|$ *is strictly convex,*

(v) if $\|\cdot\|$ *is strictly convex, the Gauss map* $G : S^{n-1}_{\|\cdot\|} \to S^{n-1}$ *is a homeomorphism.*

Proof. We begin by establishing (ii). Let $v_0 \in S^{n-1}_{\|\cdot\|}$ be a point that either maximizes or minimizes the Euclidean distance to the origin among all $v \in S^{n-1}_{\|\cdot\|}$. Let $\gamma : (-\epsilon, \epsilon) \to S^{n-1}_{\|\cdot\|}$ be a C^1-curve for which $\gamma(0) = v_0$. Thus, $\dot{\gamma}(0) \in T_{v_0} S^{n-1}_{\|\cdot\|}$, and by choice of v_0 and the product rule for derivations, it follows that $0 = \frac{\mathrm{d}}{\mathrm{d}t}\langle \gamma(t), \gamma(t)\rangle \,|_{t=0} = 2\langle \dot{\gamma}(0), \gamma(0)\rangle$. Since $G(v_0)$ is orthogonal to all $T_{v_0} S^{n-1}_{\|\cdot\|}$ it follows that $G(v_0) = \pm \frac{v_0}{|v_0|}$. Moreover, since $G(v_0)$ points outward of $S^{n-1}_{\|\cdot\|}$ at v_0, hence $G(v_0) = \frac{v_0}{|v_0|}$. This proves (ii).

In order to prove (i), let $v \in S^{n-1}_{\|\cdot\|}$ and consider $V = G(v)^{\perp} \in G(n, n-1)$. Then, the supportive hyperplane of $S^{n-1}_{\|\cdot\|}$ at v is $x + V = \{x \in \mathbb{R}^n : \langle x - v, G(v)\rangle = 0\}$. Now, assume for a contradiction that $\langle v, G(v)\rangle = 0$. Then, $0 \in x + V$. However, since $x + V$ is a supportive hyperplane of $S^{n-1}_{\|\cdot\|}$, this contradicts the fact that $S^{n-1}_{\|\cdot\|}$ bounds an antipodally symmetric set of non-empty interior. Thus, $\langle v, G(v)\rangle \neq 0$, for all $v \in S^{n-1}_{\|\cdot\|}$. Recall from the proof of (ii) that there exists a point $v_0 \in S^{n-1}_{\|\cdot\|}$ such that $G(v_0) = \frac{v_0}{|v_0|}$. Therefore, $\langle v_0, G(v_0)\rangle > 0$. Hence, (i) follows from continuity of G and the mean value theorem.

Now, consider a direction $v \in S^{n-1}$ and let V be its orthogonal complement. Since $S^{n-1}_{\|\cdot\|}$ is compact, the set $\{t > 0 : S^{n-1}_{\|\cdot\|} \cap (tv + V) \neq \varnothing\}$ has a maximum $t_0 > 0$. Thus, $H := t_0 v + V$ is the tangent plane of $S^{n-1}_{\|\cdot\|}$ at the point x where $S^{n-1}_{\|\cdot\|}$ intersects the line $L_v = \{tv : t \in \mathbb{R}\}$. Moreover, since H was chosen to be orthogonal to v, it follows that $G(x) = v$. This proves (iii).

Assume that G is injective and let $x \in S^{n-1}_{\|\cdot\|}$ and by H denote the unique supportive hyperplane of $S^{n-1}_{\|\cdot\|}$ at x. Let $y \in H \cap S^{n-1}_{\|\cdot\|}$. Now, we will deduce that $x = y$ from the assumption that G is strictly convex. Since, $x, y \in H \cap S^{n-1}_{\|\cdot\|}$, we have that $H = x + T_x S^{n-1}_{\|\cdot\|} = y + T_y S^{n-1}_{\|\cdot\|}$ and $G(x) = \pm G(y)$, hence $x = \pm y$. Assume for a contradiction that $y = -x$. Then, it follows that $x + T_x S^{n-1}_{\|\cdot\|} = -x + T_{-x} S^{n-1}_{\|\cdot\|}$ and, since $T_x S^{n-1}_{\|\cdot\|} = T_{-x} S^{n-1}_{\|\cdot\|} \in G(n, n-1)$, we have $x = -x$. However, this implies that $x = 0$ which contradicts the fact that $x \in S^{n-1}_{\|\cdot\|}$. This proves one direction of (iv).

For the converse, assume that $\|\cdot\|$ is strictly convex and let $x, y \in S^{n-1}_{\|\cdot\|}$ so that $G(x) = G(y)$. Thus, it follows that $P := T_x S^{n-1}_{\|\cdot\|} = T_y S^{n-1}_{\|\cdot\|}$. Assume without loss of generality that x and y lie on the same side of P (if they do not lie on the same side, replace x by $-x$). In case that $x + P = y + P$, strict convexity implies that $x = y$. Now, consider the case when $x + P \neq y + P$. Then, P, $x + P$ and $y + P$ are three parallel hyperplanes in $\mathbb{R}^n$ and (by the assumption that x and y lie on the same side of P) P is not the middle one. Assume that $x + P$ is the middle one (the other case is analogous). Then, $x + P$ intersects the interior of $B^n_{\|\cdot\|}$ which is a continuum connecting 0 to y. This proves (iv).

For the proof of (v), assume that $\|\cdot\|$ is a strictly convex C^1-norm on $\mathbb{R}^n$. Thus, by (ii) and (iii), $G : S^{n-1}_{\|\cdot\|} \to S^{n-1}$ is a bijection. Moreover, by (5.3) and the fact that $\|\cdot\|$ is C^1, if follows that G is continuous. Thus, since : $S^n_{\|\cdot\|}$ and S^{n-1} are both compact, G is a homeomorphism. $\square$

Remark 5.3. For a $C^{2,\delta}$-norm $\|\cdot\|$ on $\mathbb{R}^n$, $G : S^{n-1}_{\|\cdot\|} \to S^{n-1}$ is a $C^{1,\delta}$-mapping. Hence, if the Jacobian determinant of G does not vanish, and by the inverse function theorem and Lemma 5.2, $G : S^{n-1}_{\|\cdot\|} \to S^{n-1}$ is a locally injective surjection. Since $S^{n-1}_{\|\cdot\|}$ is homeomorphic to S^{n-1}, from a topological argument it follows that G is a homeomorphism. This makes $G : S^{n-1}_{\|\cdot\|} \to S^{n-1}$ a $C^{1,\delta}$-diffeomorphism. In particular, by Lemma 5.2, the norm $\|\cdot\|$ is strictly convex.

Denote the derivative of $G : S^{n-1}_{\|\cdot\|} \to S^{n-1}$ at a point x by $\mathrm{D}G(x)$. The Jacobian determinant of G at x is $\det \mathrm{D}G(x)$. In the case where $n = 2$, $S^1_{\|\cdot\|}$ is a closed C^2-curve and $\det \mathrm{D}G(x) \neq 0$ if and only if the curvature of the curve $S^1_{\|\cdot\|}$ does not vanish at x. In $\mathbb{R}^3$, the equivalent statement holds for the Gauss curvature of the C^2-surface $S^2_{\|\cdot\|}$. The Gauss curvature of $S^2_{\|\cdot\|} \subset \mathbb{R}^3$ at a point x is defined to be the determinant of the derivative of the Gauss map. In higher dimensions the analog of the Gauss curvature is called Gauss-Kronecker curvature. It is a fact that in $\mathbb{R}^n$ the determinant $\det \mathrm{D}G$ does not vanish in a point if and only if the Gauss-Kronecker curvature does not vanish in this point. Since $S^{n-1}_{\|\cdot\|}$ is convex this is equivalent to requiring all sectional curvatures in all points on $S^{n-1}_{\|\cdot\|}$ to be positive; see Chapter 6 in [10]. This discussion yields the following result.

Lemma 5.4. *For a $C^{2,\delta}$-norm $\|\cdot\|$ on $\mathbb{R}^n$, $G : S^{n-1}_{\|\cdot\|} \to S^{n-1}$ is a $C^{1,\delta}$-diffeomorphism if and only if the curvature of $S^{n-1}_{\|\cdot\|}$ does not vanish. (Here by curvature we mean the curvature of a C^2-curve if $n = 2$, the Gauss curvature of a surface if $n = 3$, and the Gauss-Kronecker or, equivalently, the sectional curvatures if $n > 3$.)*

5.2 PROJECTION THEOREMS FOR CODIMENSION ONE

In this section we will verify that given a sufficiently regular norm $\|\cdot\|$ on $\mathbb{R}^n$, the family of projections onto hyperplanes $V \in G(n, n-1)$ induced by $\|\cdot\|$ is a family of linear and surjective projections. Moreover, its associated map $\mathscr{G} : G(n, n-1) \to G(n, n-1)$ (see (4.1) and Remark 4.4), can be expressed in terms of the inverse Gauss map of $\|\cdot\|$. This will allow us to prove the following theorem.

Theorem 5.5. *Let $\|\cdot\|$ be a strictly convex C^1-norm on $\mathbb{R}^n$. If the Gauss map G is dimension non-increasing and has the Lusin property, then conclusions (1) and (2) of Theorem 4.2 hold for $P^{\|\cdot\|} : G(n, n-1) \times \mathbb{R}^n \to \mathbb{R}^n$.*

Note that we do not mention conclusion (3) of Theorem 4.2 in Theorem 5.5 since $m = n - 1$ and conclusion (3) only makes sense when $2m \leq n$.

We will prove Theorem 5.5 by applying Theorem 4.2. For this, first, we need to establish that families of projections $P^{\|\cdot\|}$ that meet the conditions of Theorem 5.5 are families of linear and surjective projections. Moreover, we want to find an injective mapping $\widetilde{\mathscr{G}} : S^{n-1} \to S^{n-1}$ for which $\widetilde{\mathscr{G}}(v) \in \operatorname{Ker} P^{\|\cdot\|}_{v^\perp}$ for all $v \in S^{n-1}$; see Remark 4.4.

Lemma 5.6. *For a strictly convex C^1-norm $\|\cdot\|$, the map $P^{\|\cdot\|} : G(n, n-1) \times \mathbb{R}^n \to \mathbb{R}^n$ is a family of linear and surjective projections. Moreover, the map $\widetilde{\mathscr{G}} : S^{n-1} \to S^{n-1}$ defined by*

$$\widetilde{\mathscr{G}}(w) = \frac{G^{-1}(w)}{|G^{-1}(w)|},$$

for all $w \in S^{n-1}$, is an injective mapping for which $\widetilde{\mathscr{G}}(v) \in \operatorname{Ker} P_{v^\perp}$ for all $v \in S^{n-1}$.

Proof. Let $V \in G(n, n-1)$. First, recall that for all $x \in \mathbb{R}^n \backslash V$, $P_V^{\|\cdot\|}(x)$ is the unique point in the intersection $S^{n-1}_{\|\cdot\|}(x, \operatorname{dist}_{\|\cdot\|}(x, V)) \cap V$. Therefore, V must be the tangent plane of $S^{n-1}_{\|\cdot\|}(x, \operatorname{dist}_{\|\cdot\|}(x, V))$ at $P_V^{\|\cdot\|}(x)$.

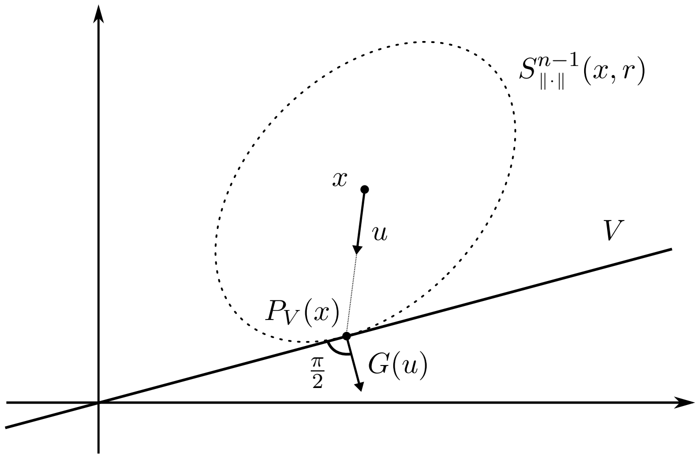

Figure 5.2. Gauss map and projections $\left(u = \frac{P_V^{\|\cdot\|}(x) - x}{\|P_V^{\|\cdot\|}(x) - x\|} \text{ and } r = \operatorname{dist}_{\|\cdot\|}(x, V)\right)$.

However, this implies that the unit outward normal of $S^{n-1}_{\|\cdot\|}(x, \operatorname{dist}_{\|\cdot\|}(x, V))$ at $P_V^{\|\cdot\|}(x)$ is orthogonal to V, or, equivalently (see Figure 5.2),

$$G\left(\frac{P_V^{\|\cdot\|}(x) - x}{\|P_V^{\|\cdot\|}(x) - x\|}\right) \perp V.$$

Let $w = w(V) \in S^{n-1}$ be a direction that is orthogonal to V, then

$$G\left(\frac{P_V^{\|\cdot\|}(x) - x}{\|P_V^{\|\cdot\|}(x) - x\|}\right) = \lambda w,$$

where $\lambda \in \{-1, 1\}$. Using the fact that G is invertible and antipodally symmetric, this yields that

$$\frac{P_V^{\|\cdot\|}(x) - x}{\|P_V^{\|\cdot\|}(x) - x\|} = \lambda G^{-1}(w). \tag{5.4}$$

Thus, for every $x \in \mathbb{R}^n$, the vector $P_V^{\|\cdot\|}(x) - x$ is collinear with $G^{-1}(w)$. Moreover, by (i) and (v) of Lemma 5.2, $G^{-1}(w)$ is not contained in V. Hence, $P_V^{\|\cdot\|}(x)$ is the unique intersection point of the line $x + L_{G^{-1}(w)}$ with the m-plane V (recall that $L_v := \{rv : r \in \mathbb{R}\}$ for all $v \in \mathbb{R}^n \backslash \{0\}$). This proves that $P_V^{\|\cdot\|} : \mathbb{R}^n \to V$ a linear map. To see this, choose a basis $\{b_1, \ldots, b_n\}$ or $\mathbb{R}^n$ where b_1 is collinear with $L_{G^{-1}(w)}$ and the vectors $b_2, \ldots, b_n$ form a basis of V. Then, $P_V : \mathbb{R}^n \to V$ is given by $P_V(x) = x_2 b_2 + \ldots + b_n x_n$ for all $x \in \mathbb{R}^n$ where the x_i are the coefficients of x in the basis $b_1, \ldots, b_n$, i.e. $x = x_1 b_1 + \ldots + x_n b_n$. Furthermore, it follows that $(P_V^{\|\cdot\|})^{-1}(\{0\}) = L_{G^{-1}(w)}$, and thus, $\mathscr{G}(V) = (G^{-1}(w))^{\perp} = \big(\frac{G^{-1}(w)}{|G^{-1}(w)|}\big)^{\perp}$. □

Proof of Theorem 5.5. By Lemma 5.6, Theorem 4.2, and Remark 4.4, it suffices to check that the mapping $\widetilde{\mathscr{G}} : S^{n-1} \to S^{n-1}$ defined by $\widetilde{\mathscr{G}}(w) = \frac{G^{-1}(w)}{|G^{-1}(w)|}$ is dimension non-decreasing and has the inverse Lusin property. From the fact that any two norms on $\mathbb{R}^n$ are bi-Lipschitz equivalent, in particular, it follows that

$$h : S^{n-1}_{\|\cdot\|} \to S^{n-1}_{|\cdot|} = S^{n-1}$$

given by $h(x) = \frac{x}{|x|}$ for all $x \in S^{n-1}_{\|\cdot\|}$ is a bi-Lipschitz mapping.
Note that $\widetilde{\mathscr{G}} = h \circ G^{-1}$ and hence, $\widetilde{\mathscr{G}}$ is dimension non-decreasing and has the inverse Lusin property, if and only if G^{-1} is dimension non-decreasing and has the inverse Lusin property. However, this is guaranteed by the assumption that G is dimension non-increasing and has the Lusin property. □

The following corollary is a simple consequence of the proof of Theorem 5.5 and the fact that Lipschitz mappings are dimension non-increasing.

Corollary 5.7. *If $\|\cdot\|$ is a strictly convex $C^{1,1}$-Norm on $\mathbb{R}^n$, then conclusions (1) and (2) of Theorem 4.2 hold.*

Moreover, from Theorem 4.5 and the proof of Theorem 5.5 one immediately deduces the following corollary.

Corollary 5.8. *Let $\|\cdot\|$ be a strictly convex C^1-norm on $\mathbb{R}^n$ such that its Gauss map G has the Lusin property as well as the inverse Lusin property. Then, a set $A \subseteq \mathbb{R}^n$ with $\mathscr{H}^m(A) < \infty$ is purely $(n-1)$-unrectifiable if and only if $\mathscr{H}^{n-1}(P_V^{\|\cdot\|}(A)) = 0$ for $\sigma_{n,(n-1)}$-a.e. $V \in G(n,m)$.*

5.3 TRANSVERSALITY FOR CODIMENSION ONE

In this section, we will basically reprove Theorem 5.5 for sufficiently regular norms by establishing differentiable transversality; see Definition 3.9 and Theorem 3.11. These results can also be found in [3] for the case $n = 2$. We think that this proof is worth being included in this thesis for several reasons. First, it provides an insightful example of how differentiable transversality can be proven in a specific setting. Second, it illustrates the limits of transversality as a method of proof for Marstrand-type projection theorems as we will see that the Marstrand-type results that we obtain here will be weaker than Theorem 5.5 obtained by comparison. Third, it shows that metric transversality and differentiable transversality are not equivalent; see Corollary 5.14. And last, as pointed out in the introduction, differentiable transversality is a notion studied for many families of mappings (not a priori families of projections in a geometric sense) in different areas of mathematics. This makes it a property of independent interest.

Let $\|\cdot\|$ be a strictly convex norm on $\mathbb{R}^n$ and by $P^{\|\cdot\|} : G(n, n-1) \times \mathbb{R}^n \to \mathbb{R}^n$ denote the family of closest-point projections with respect to $\|\cdot\|$ as defined in (5.2). Let $\Pi^{\|\cdot\|} : Q \times \Omega \to \mathbb{R}^{n-1}$ be the associated family of abstract projections onto $\mathbb{R}^{n-1}$ with $Q = \mathrm{Mat}_{1\times(n-1)}(\mathbb{R})$ and $\Omega \subset \mathbb{R}^n$ a large ball centered at the origin; see (3.8).

Theorem 5.9. *Let $\delta > 0$ and consider a $C^{2,\delta}$-norm $\|\cdot\|$ on $\mathbb{R}^n$ for which $\det \mathrm{D}G(v) \neq 0$ for all $v \in S^{n-1}_{\|\cdot\|}$. Then, the family of projections $\Pi^{\|\cdot\|} : Q \times \Omega \to \mathbb{R}^{n-1}$ satisfies differentiable transversality for $L = 1$ and $\delta > 0$ and therefore the conclusions of Theorem 3.11 hold for $\Pi^{\|\cdot\|}$ with $L = 1$.*

Recall from Section 5.1 that the assumption $\det \mathrm{D}G(v) \neq 0$ for all $v \in S^{n-1}_{\|\cdot\|}$ guarantees that $\|\cdot\|$ is strictly convex and thus the family $\Pi^{\|\cdot\|} : Q \times \Omega \to \mathbb{R}^{n-1}$ is well-defined.

The following corollary is a straight-forward consequence of Theorem 5.9 and Lemma 5.4.

Corollary 5.10. *Let $\delta > 0$ and consider a $C^{2,\delta}$-norm $\|\cdot\|$ on $\mathbb{R}^n$ such that $S^{n-1}_{\|\cdot\|}$ has non-zero sectional curvature. Then, the family of projections $\Pi^{\|\cdot\|} : Q \times \Omega \to \mathbb{R}^{n-1}$ that satisfies differentiable transversality with constants $L = 1$ and $\delta > 0$, and therefore the conclusions of Theorem 3.11 hold for $\Pi^{\|\cdot\|}$ for $L = 1$ and the respective $\delta > 0$.*

The proof of Theorem 5.9, will be divided into a sequence of lemmas. For the sake of generality, we will state and prove some of these lemmas under slightly weaker assumptions than necessary for the proof of Theorem 5.9. In the first lemma, we exploit the arguments from the proof of Lemma 5.6 in order to obtain an explicit formula for the projection $P^{\|\cdot\|}_V : \mathbb{R}^n \to V$.

Lemma 5.11. *Let $\|\cdot\|$ be a strictly convex C^1-norm and $G : S^{n-1}_{\|\cdot\|} \to S^{n-1}$ the Gauss map associated with $\|\cdot\|$. Then for every $x \in \mathbb{R}^n$ and $V \in G(n, n-1)$*

$$P^{\|\cdot\|}(V, x) = x - \frac{\langle x, w\rangle}{\langle G^{-1}(w), w\rangle}\, G^{-1}(w), \tag{5.5}$$

where $w = w(V) \in S^{n-1}$ is orthogonal to V.

Proof. Consider $V \in G(n, m)$ and $w \in S^{n-1}$ a direction orthogonal to V. Then, by (5.4), it follows that

$$P^{\|\cdot\|}_V(x) = x + \|P^{\|\cdot\|}_V(x) - x\|\, G^{-1}(\lambda w), \tag{5.6}$$

for all $x \in \mathbb{R}^n$ where $\lambda \in \{-1, 1\}$ depends on the position of x. More precisely, $\lambda = 1$ if $\langle x, w \rangle \leq 0$, and $\lambda = -1$ if $\langle x, w \rangle > 0$.
On the other hand, by choice of w and the fact that $P_V^{\|\cdot\|}(x) \in V$ for all $x \in \mathbb{R}^n$,

$$\langle P_V^{\|\cdot\|}(x), w \rangle = 0. \tag{5.7}$$

Then, by (5.6) and (5.7), it follows that

$$\|P_V^{\|\cdot\|}(x) - x\| = -\frac{\langle x, w \rangle}{\langle G^{-1}(\lambda w), w \rangle} = -\lambda \frac{\langle x, w \rangle}{\langle G^{-1}(w), w \rangle} \tag{5.8}$$

Finally, combining (5.6) with (5.8) yields the desired projection formula

$$P_V^{\|\cdot\|}(x) = x - \frac{\langle x, w \rangle}{\langle G^{-1}(\lambda w), w \rangle}\, G^{-1}(\lambda w) = x - \frac{\langle x, w \rangle}{\langle G^{-1}(w), w \rangle}\, G^{-1}(w).$$

□

Notice that Lemma 5.6 is a trivial consequence of Lemma 5.11. However, we decided not to prove Lemma 5.11 in Section 5.2 in order to stress that the explicit formula for the family of projections given in Lemma 5.11 is not required for the proof of Theorem 5.5. However, for our proof of Theorem 5.9, Lemma 5.11 will be essential.

The following lemma is the key tool in order to establish property (P3) from Proposition 4.7 for the family of projections $\Pi^{\|\cdot\|} : Q \times \Omega \to \mathbb{R}^{n-1}$.

Lemma 5.12. *Let $\|\cdot\|$ be a C^2- norm on $\mathbb{R}^n$ such that* $\det \mathrm{D}G(x) \neq 0$ *for all $x \in S^{n-1}_{\|\cdot\|}$. Let $x \in \mathbb{R}^n \backslash \{0\}$ and $V_0 \in G(n, n{-}1)$ such that $P^{\|\cdot\|}(V_0, x) = 0$. Then, the differential $\mathrm{D}_V P^{\|\cdot\|}(V_0, x) : T_{V_0} G(n, n{-}1) \to V_0$ is an isomorphism.*

Proof. It suffices to show that $\mathrm{D}_V P^{\|\cdot\|}(V_0, x)(u) \subseteq (V_0 \backslash \{0\})$ for all tangent vectors $u \in T_{V_0} G(n, n{-}1) \backslash \{0\}$. Let $u \in T_{V_0} G(n, n{-}1) \backslash \{0\}$ and $\gamma : (-\epsilon, \epsilon) \to G(n, n{-}1)$ a smooth curve such that $\gamma(0) = V_0$, $\dot{\gamma}(0) = u$. Now, choose $\beta : (-\epsilon, \epsilon) \to S^{n-1}$ to be a smooth curve such that $\beta(s) \in S^{n-1}$ is orthogonal to $\gamma(s) \in G(n, n{-}1)$ for all $s \in (-\epsilon, \epsilon)$. Without loss of generality, we assume that β is parameterized by arc-length. Recall from Remark 5.3 that the assumption that $\det \mathrm{D}G(x) \neq 0$ for all $x \in S^{n-1}_{\|\cdot\|}$, implies that the Gauss map $G : S^{n-1}_{\|\cdot\|} \to S^{n-1}$ is a C^1-diffeomorphism, and in particular, $\|\cdot\|$ is C^1. Define the mapping $\psi : (-\epsilon, \epsilon) \to \mathbb{R}$ by

$$\psi(s) := \frac{\langle x, \beta(s) \rangle}{\langle G^{-1}(\beta(s)), \beta(s) \rangle}.$$

Then, by (5.5), it follows that

$$P^{\|\cdot\|}(\gamma(s), x) = x - \psi(s)\, G^{-1}(\beta(s)) \tag{5.9}$$

and thus

$$
\begin{aligned}
\mathrm{D}_V P^{\|\cdot\|}(V_0, x)(u) &= \frac{\mathrm{d}}{\mathrm{d}s} P^{\|\cdot\|}(\gamma(s), x)|_{s=0} \\
&= \left(-\dot{\psi}(s) G^{-1}(\beta(s)) - \psi(s)\,\mathrm{D}G^{-1}(\beta(s))(\dot{\beta}(s))\right)\Big|_{s=0} \\
&= -\dot{\psi}(0) G^{-1}(\beta(0)) - \psi(0)\,\mathrm{D}G^{-1}(\beta(0))(\dot{\beta}(0)).
\end{aligned}
\tag{5.10}
$$

Note that $\mathrm{D}G^{-1}(w)$ denotes the differential of the inverse Gauss map $G^{-1} : S^{n-1} \to S^{n-1}_{\|\cdot\|}$ at a point $w \in S^{n-1}$; see Figure 5.3.

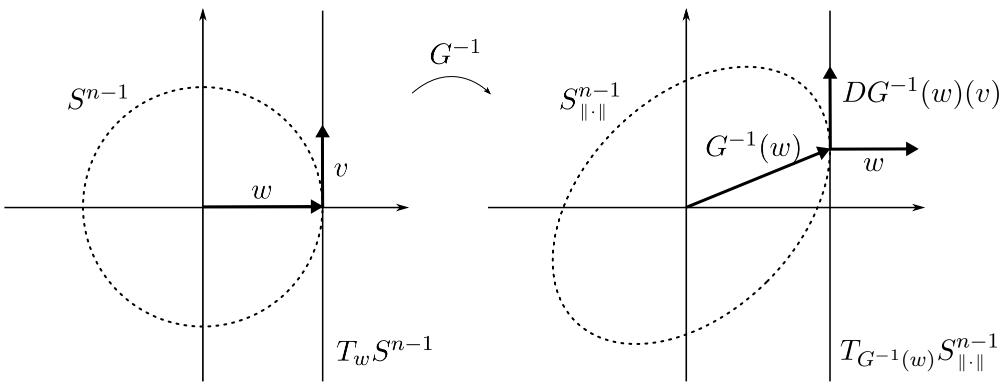

Figure 5.3. The derivative of the inverse Gauss map.

Thus, by definition, $\mathrm{D}G^{-1}(\beta(0))(\dot{\beta}(0)) \in T_{G^{-1}(\beta(0))} S^{n-1}_{\|\cdot\|}$. However, by definition of the Gauss map G, $\beta(0)$ is orthogonal to $T_{G^{-1}(\beta(0))} S^{n-1}_{\|\cdot\|}$ and hence, $V_0 = T_{G^{-1}(\beta(0))} S^{n-1}_{\|\cdot\|}$. Moreover, by the assumption that G^{-1} is a C^1-diffeomorphism and the fact that $\dot{\beta}(0) \neq 0$, it follows that

$$
\mathrm{D}G^{-1}(\beta(0))(\dot{\beta}(0)) \in V_0 \backslash \{0\}. \tag{5.11}
$$

Now, by (5.10) and (5.11), it suffices to check that $\psi(0) \neq 0$ and $\dot{\psi}(0) = 0$.

Since $x \neq 0$ and $P^{\|\cdot\|}(\gamma(0), x) = 0$, by (5.9), $\psi(0) \neq 0$. Now, for $s \in (-\epsilon, \epsilon)$, $\dot{\psi}(s)$ equals

$$
\frac{\langle \dot{\beta}(s), x\rangle \langle \beta(s), G^{-1}(\beta(s))\rangle - \langle \beta(s), x\rangle \left[\langle \dot{\beta}(s), G^{-1}(\beta(s))\rangle + \langle \beta(s), \mathrm{D}G^{-1}(\beta(s))(\dot{\beta}(s))\rangle\right]}{\langle \beta(s), G^{-1}(\beta(s))\rangle^2}.
$$

Since $\mathrm{D}G^{-1}(\beta(s))(\dot{\beta}(s)) \in T_{G^{-1}(\beta(s))} S^{n-1}_{\|\cdot\|}$ and $\beta(s)$ is orthogonal to $T_{G^{-1}(\beta(s))} S^{n-1}_{\|\cdot\|}$, for all s, it follows that $\beta(s) \cdot \mathrm{D}G^{-1}(\beta(s))(\dot{\beta}(s)) = 0$, and hence

$$
\dot{\psi}(s) = \frac{\langle \dot{\beta}(s), x\rangle \langle \beta(s), G^{-1}(\beta(s))\rangle - \langle \beta(s), x\rangle \langle \dot{\beta}(s), G^{-1}(\beta(s))\rangle}{\langle \beta(s), G^{-1}(\beta(s))\rangle^2} \tag{5.12}
$$

for all $s \in (-\epsilon, \epsilon)$.

Recall that $P^{\|\cdot\|}(\gamma(0),x) = x - \psi(0)G^{-1}(\beta(0))$ and that $\psi(0) \neq 0$. Thus, it follows that $G^{-1}(\beta(0)) = \frac{1}{\psi(0)}x$. Finally, plugging this into (5.12) yields $\dot{\psi}(0) = 0$. $\square$

Lemma 5.13. *Let $\|\cdot\|$ be a C^2-norm on $\mathbb{R}^n$ such that $\det \mathrm{D}G(v) \neq 0$ for all $v \in S^{n-1}_{\|\cdot\|}$. Then, the family of projections $\Pi^{\|\cdot\|} : Q \times \Omega \to \mathbb{R}^{n-1}$ satisfies properties (P1), (P2) and (P3).*

Proof. Define $T \mapsto w_T$ to be a smooth mapping $\mathrm{Mat}_{1\times(n-1)}(\mathbb{R}) \to S^{n-1}$ such that w_T is orthogonal to V_T for all $T \in \mathrm{Mat}_{1\times(n-1)}(\mathbb{R})$. (For this, we combine the parameterization $\varphi : Q \to G(n,n-1)$, $\varphi(T) = V_T$, with the identification of $G(n,n-1)$ with S^{n-1}; see Section 2.3.) Moreover, by Remark 5.3, the Gauss map $G : S^{n-1}_{\|\cdot\|} \to S^{n-1}$ is a $C^{1,\delta}$-diffeomorphism. Thus, by (5.5),

$$P^{\|\cdot\|}(V_T,x) = x - \frac{\langle x, w_T\rangle}{\langle G^{-1}(w_T), w_T\rangle}\, G^{-1}(w_T),$$

for all $T \in \mathrm{Mat}_{1\times(n-1)}(\mathbb{R})$ and $x \in \mathbb{R}^n$.

Recall from (3.8) that, for $T \in \mathrm{Mat}_{1\times(n-1)}(\mathbb{R})$ and $x \in \Omega$,

$$\Pi^{\|\cdot\|}(T,x) = \sum_{i=1}^{n-1} \left\langle P^{\|\cdot\|}(V_T,x), e_i^T \right\rangle w_i.$$

Thus, since G is a C^1-diffeomorphism, $T \mapsto \Pi^{\|\cdot\|}(T,x)$ is of class C^1 for all $x \in \Omega$. This proves (P1).

Now, from continuity of $\Pi^{\|\cdot\|}$ and $\mathrm{D}_T\Pi^{\|\cdot\|}$, it follows that $\Pi^{\|\cdot\|}$ and $\mathrm{D}_T\Pi^{\|\cdot\|}$ are bounded on $Q' \times \Omega$ for all $Q' \subset \mathrm{Mat}_{1\times(n-1)}(\mathbb{R})$ compact. Then, since G is a $C^{1,\delta}$-diffeomorphism, G^{-1} is a $C^{1,\delta}$-mapping and thus, $T \mapsto \Pi^{\|\cdot\|}(T,x)$ is of class $C^{1,\delta}$. Recall from (2.3) that here $T = (t_1,\dots,t_{n-1})$. Thus, for all $j = 1,\dots,n-1$, $T \mapsto \frac{\partial}{\partial t_j}\Pi^{\|\cdot\|}(T,x)$ is δ-Hölder on $Q' \times \Omega$ for all $Q' \subset Q$ compact. This proves (P2).

For the proof (P3), fix some compact and connected set Q' in Q, and let $x \in \mathbb{R}^n$ and $T_0 \in Q'$ such that $\Pi^{\|\cdot\|}(T_0,x) = 0$. The product rule for derivations yields that the i,j-th entry $[\mathrm{D}_T\Pi^{\|\cdot\|}(T_0,x)]_{i,j}$ of the matrix $\mathrm{D}_T\Pi^{\|\cdot\|}(T_0,x)$ is

$$\begin{aligned}
[\mathrm{D}_T\Pi^{\|\cdot\|}(T_0,x)]_{i,j} &= \frac{\partial}{\partial t_j} \left\langle P^{\|\cdot\|}(\varphi(T),x), e_i^T \right\rangle \|_{T=T_0} \\
&= \left\langle \mathrm{D}_V P^{\|\cdot\|}(\varphi(T_0),x)\Big(\frac{\partial}{\partial t_j}\varphi(T_0)\Big), e_i^{T_0} \right\rangle + \left\langle P^{\|\cdot\|}(\varphi(T_0),x), \frac{\partial}{\partial t_j} e_i^{T_0} \right\rangle.
\end{aligned}$$

However, by assumption $P^{\|\cdot\|}(\varphi(T_0),x) = 0$ and hence

$$[\mathrm{D}_T\Pi^{\|\cdot\|}(T_0,x)]_{i,j} = \left\langle \mathrm{D}_V P^{\|\cdot\|}\Big(\varphi(T_0),x\Big)\Big(\frac{\partial}{\partial t_j}\varphi(T_0)\Big), e_i^{T_0} \right\rangle.$$

Recall the following fact from linear algebra: Let A be an invertible $m \times m$-matrix over $\mathbb{R}$, $\{w_i\}_{i=1}^m$ a basis of $\mathbb{R}^m$, and let $\{v_i\}_{i=1}^m$ be an orthonormal basis of $\mathbb{R}^m$. By $\widetilde{A}$ denote the $(m \times m)$-matrix whose (i,j)-th entry equals $\langle Aw_i, v_j \rangle$. Then, the rows of $\widetilde{A}$ are the vectors Aw_i, $i = 1, \dots, m$, represented in the basis $\{v_i\}_{i=1}^m$ and thus, $\widetilde{A}$ is invertible.

Hence, since the vectors $e_i^{T_0}$ form an orthonormal basis of V_{T_0} (see Section 2.3), it follows that $\mathrm{D}_T \Pi^{\|\cdot\|}(T_0, x)$ is invertible and $\det \mathrm{D}_T \Pi(T_0, x) \neq 0$. $\square$

Proof of Theorem 5.9. Let $\delta > 0$ and consider a $C^{2,\delta}$-regular norm $\|\cdot\|$ on $\mathbb{R}^n$ whose Gauss map $G : S^{n-1}_{\|\cdot\|} \to S^{n-1}$ satisfies $\det \mathrm{D}G(v) \neq 0$ for all $v \in S^{n-1}_{\|\cdot\|}$. Recall from Section 5.1 that this makes $G : S^{n-1}_{\|\cdot\|} \to S^{n-1}$ a $C^{1,\delta}$-diffeomorphism.
Now, we apply Lemmas 5.11, 5.12 and 5.13, as well as Proposition 4.7. This yields that for all $R \subset Q$ open and compactly contained in Q, the family $\Pi^{\|\cdot\|} : R \times \Omega \to \mathbb{R}^m$ satisfies differentiable transversality with constants $L = 1$ and $\delta > 0$. Thus, since $G(n, n-1)$ is compact, all the constants in Definition 3.9 can be chosen independently of R. Thus, $\Pi^{\|\cdot\|} : Q \times \Omega \to \mathbb{R}^m$ satisfies differentiable transversality with $L = 1$ and $\delta > 0$. $\square$

There are two aspects in which Theorem 5.9 is weaker than Theorem 5.5. First, Theorem 5.5 implies that the conclusions of Theorem 3.11 hold for $\Pi^{\|\cdot\|}$ with $L = \infty$, while Theorem 5.9 only implies the conclusions for $L = 1$. Second, Theorem 5.9 requires the norm to be a $C^{2,\delta}$-norm while Theorem 5.5 requires $C^{1,1}$ only. It is obvious from Definition 3.9 and Lemma 5.11 that any weaker regularity than $C^{2,\delta}$ will not suffice for the proof of Theorem 5.9.

The following corollary is a direct consequence of Theorem 5.5 and (the proof of) Theorem 5.9.

Corollary 5.14. *For a norm $\|\cdot\|$ that is $C^{1,1}$ but not C^2, the conclusions of Theorem 3.11 as well as metric transversality (Definition 3.2) hold, however, $\Pi^{\|\cdot\|}$ does not satisfy differentiable transversality (Definition 3.9).*

Combining Theorem 5.9 with Theorem 3.14 immediately implies the following: Let $\|\cdot\|$ be a $C^{3,0}$-norm on $\mathbb{R}^n$ for which $\det \mathrm{D}G(v) \neq 0$ for all $v \in S^{n-1}_{\|\cdot\|}$ and let $A \subseteq \Omega$ with $\mathscr{H}^m(A) < \infty$. Then, A is purely $(n-1)$-unrectifiable if and only if $\mathscr{H}^{n-1}(\Pi_T^{\|\cdot\|}(A)) = 0$ for $\mathscr{H}^{n-1}$-a.e. $T \in \mathrm{Mat}_{1\times(n-1)}(\mathbb{R})$. However, this is not the strongest possible version of a Besicovich-Federer projection theorem in this setting. Namely, by combining Theorem 4.5 and Remark 5.3, we obtain the following corollary.

Corollary 5.15. *Let $\|\cdot\|$ be a $C^{2,1}$-norm on $\mathbb{R}^n$ for which $\det \mathrm{D}G(v) \neq 0$ for all $v \in S^{n-1}_{\|\cdot\|}$ and let $A \subseteq \Omega$ with $\mathscr{H}^m(A) < \infty$. Then, A is purely $(n-1)$-unrectifiable if and only if $\mathscr{H}^{n-1}(\Pi_T^{\|\cdot\|}(A)) = 0$ for $\mathscr{H}^{n-1}$-a.e. $T \in \mathrm{Mat}_{1\times(n-1)}(\mathbb{R})$.*

As described in Section 5.5.3, it is easy to generate families of linear and surjective projections for which Marstrand-type projection theorem fails. Similar examples are obtained from norms for which the Gauss map is not defined or multivalued for some points; see Figures 4 and 6 in [3]. This raises the natural question, whether there exists a C^1-norm on $\mathbb{R}^n$ for which Marstrand-type projection theorems fail. In this section, we will construct such a norm on $\mathbb{R}^2$; see Theorem 5.16.

The following theorem states that there exist C^1-norms on $\mathbb{R}^2$ for which Marstrand-type projection theorems fail. This result underlines the relevance of Theorem 5.5 and Corollary 5.7.

Theorem 5.16. *There exists a strictly convex C^1-norm on $\mathbb{R}^2$ such that conclusion (1) of Theorem 4.2 fails for the family of projections $P^{\|\cdot\|} : G(2,1) \times \mathbb{R}^2 \to \mathbb{R}^2$.*

For the proof of Theorem 5.16 we will explicitly construct a norm for which conclusion (1) of Theorem 4.2 fails. Furthermore, we will see that by an analogous construction one obtains a norm for which conclusion (2) of Theorem 4.2 does not hold; see the remarks after the proof of Theorem 5.16 for this.

Recall from the proofs of Theorems 4.2 and 5.5 that given a Borel set $A \subseteq \mathbb{R}^n$, if the Gauss map of a norm $\|\cdot\|$ does not blow up the $\mathscr{H}^1$-measure and dimension of the exceptional set $E \subset S^{n-1}$ of the family of Euclidean projections, then conclusion (1) of Theorem 4.2 holds. By the same argument, one can see that if $\dim E < 1$ and the Gauss map of a norm does blow up E to a set of positive $\mathscr{H}^1$-measure, then conclusion (1) of Theorem 4.2 fails. Therefore, in order to prove Theorem 5.16, we need to construct a norm $\|\cdot\|$ on $\mathbb{R}^2$ that blows up some small exceptional set E of the family of Euclidean projections to a set of positive $\mathscr{H}^1$-measure. As pointed out in the introduction, very little is known about the structure of the exceptional sets E and it is therefore not sufficient to find a norm whose Gauss map fails to not increase Hausdorff measure and dimension in general. We need the Gauss map to increase the dimension of an exceptional set. This makes the proof of Theorem 5.16 a non-trivial matter.

The following lemmas will be used in the proof of Theorem 5.16.

Lemma 5.17. *Consider an interval $I \subset \mathbb{R}$ and two continuous curves $\alpha : I \to \mathbb{R}^m$ and $\beta : I \to \mathbb{R}^n$. Suppose that there exists a constant $M > 0$ for which*

$$|\beta(s) - \beta(s')| \leq M|\alpha(s) - \alpha(s')|, \tag{5.13}$$

for all $s, s' \in I$. Then, for all Borel sets $F \subseteq [0,1]$ and for all $t > 0$,

$$\mathscr{H}^t(\beta(F)) \leq (2M)^t \mathscr{H}^t(\alpha(F)). \tag{5.14}$$

In particular, if follows that if $\mathscr{H}^1(\beta(F)) > 0$, then $\mathscr{H}^1(\alpha(F)) > 0$.

We prove Lemma 5.17 and applying a simple covering argument using the definition of the Hausdorff t-measure $\mathscr{H}^t$.

Proof. Let $t > 0$ and $F \subseteq I$ a Borel set. In the case when $\mathscr{H}^t(\alpha(F)) = \infty$, (5.14) holds trivially. Therefore, we assume that $\mathscr{H}^t(\alpha(F)) = c$ where $0 \leq c < \infty$. Let $\delta > 0$. Then, there exists an open covering $\mathcal{A} := \{A_i\}_{i=1}^N$ of $\alpha(F)$ where $N \in \mathbb{N} \cup \{\infty\}$ for which $\operatorname{diam} A_i \leq \delta$, for all $i = 1, \ldots, N$ and $\sum_{i=1}^N \operatorname{diam} A_i^t \leq c + \delta$. Without loss of generality, assume that $A_i \cap \alpha(F) \neq \varnothing$ for all $i = 1, \ldots, N$. Let $s_i \in I$ such that $\alpha(s_i) \in A_i \cap \alpha(F)$. Then, by (5.13), the family of closed balls B_i with center $\beta(s_i)$ and radius $M \operatorname{diam} A_i$ covers $\beta(F)$ and $\operatorname{diam} B_i = 2M \operatorname{diam} A_i \leq 2M\delta$ for all $i = 1, \ldots, N$. This yields

$$\mathscr{H}^t_{2M\delta}(\beta(F)) \leq \sum_{i=1}^N (\operatorname{diam} B_i)^t \leq (2M)^t \sum_{i=1}^N (\operatorname{diam} A_i)^t \leq (2M)^t(c + \delta),$$

and hence $\mathscr{H}^t((\beta(I)) \leq (2M)^t\, c$. □

The following lemma is an application of Lemma 5.17.

Lemma 5.18. *Let $b \in (0, \infty]$ and let $f, g : [0, b] \to [0, \infty)$ be two strictly increasing functions . Define $h(t) := f(t)g(t)$ for all $t \in [0, b]$. Then, for all Borel sets $F \subseteq [0, b]$, if $\mathscr{H}^1(f(F)) > 0$, then $\mathscr{H}^1(h(F)) > 0$.*

Proof. Let $F \subseteq [0, b]$ be a Borel set with $\mathscr{H}^1(f(F)) > 0$. Then, by sub-additivity of $\mathscr{H}^1$ and the fact that f is increasing, there exists a number $n \in \mathbb{N}$ with $n > \frac{1}{b}$, such that for $F_n := F \cap [\frac{1}{n}, b]$, we have $\mathscr{H}^1(f(F_n)) > 0$. For $s < s' \in [\frac{1}{n}, b]$, we have

$$h(s') - h(s) = f(s')g(s') - f(s)g(s) \geq (f(s') - f(s))g(s') \geq g(\tfrac{1}{n})f(s) - f(s') > 0.$$

Applying Lemma 5.17 for $\alpha = f : [\frac{1}{n}, b] \to [0, \infty)$, $\beta = h : [\frac{1}{n}, b] \to [0, \infty)$, and $M = \frac{1}{g(\frac{1}{n})}$, yields $\mathscr{H}^1(h(F)) \geq \mathscr{H}^1(h(F_n)) > 0$. □

Furthermore, the proof of Theorem 5.16 uses an adapted version of the devil's staircase function that we introduce in the following remark.

Remark 5.19. Let K be the triadic Cantor set, i.e. the set that is obtained by removing the middle third of the interval $[0, 1]$ and then inductively removing the middle third of each remaining interval. More formally, K is the invariant set of the iterated function system $\mathcal{S} = \{S_1, S_2\}$ where $S_i : \mathbb{R} \to \mathbb{R}$ given by $S_1(t) = \frac{t}{3}$ and $S_2(t) = \frac{2}{3} + \frac{t}{3}$; see [30]. Then, K is a set of Hausdorff dimension $s := \frac{\log(2)}{\log(3)}$ and $0 < \mathscr{H}^s(K) < \infty$.

Set $M = \mathscr{H}^s(K) > 0$ and define the triadic Cantor function $g : [0, 1] \to [0, 1]$ by $g(t) = \frac{1}{M}\mathscr{H}^s(K \cap [0, 1])$; see left-hand side of Figure 5.4. Then, g is non-decreasing and, since $\mathscr{H}^s$ does not assign mass to single points, g is continuous and surjective. Moreover the image of $[0, 1]\backslash K$ under g consists of countably many points and hence $\mathscr{H}^1(g([0, 1]\backslash K)) = 0$ and $\mathscr{H}^1(g(K)) = 1$.

In the sequel, we will need a function f that has similar measure theoretic properties as g but is strictly increasing. We can construct such a function as follows. Let $f : [0,1] \to [0,1]$ be defined by $f(t) := \frac{1}{2}(g(t)+t)$; see right-hand side of Figure 5.4.

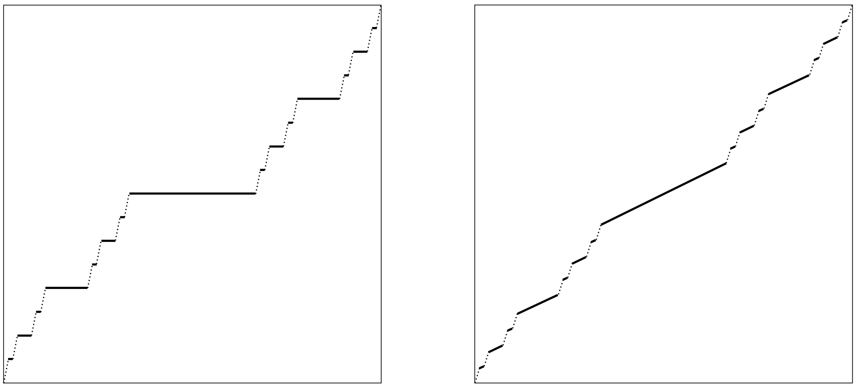

Figure 5.4. The triadic Cantor function and an injective variant.

Then, f is strictly increasing, continuous and surjective, and hence a homeomorphism. Since g is constant on each interval I that is contained in $[0,1]\backslash K$, g maps this interval to an interval of half its length. Then, since $[0,1]\backslash K$ consists of countably many open intervals, $\mathscr{H}^1(f([0,1]\backslash K)) = \frac{1}{2}$, and hence $\mathscr{H}^1(f(K)) = \frac{1}{2}$.

Proof of Theorem 5.16. Notice that in order to prove Theorem 5.16, it suffices to construct a norm $\|\cdot\|$ on $\mathbb{R}^2$ and a Borel set $A \subset \mathbb{R}^2$ with $\dim A = d < 1$ for which

$$\mathscr{H}^1(\{w \in S^1 : \dim P^{\|\cdot\|}_{w^\perp}(A) < d\}) > 0. \tag{5.15}$$

Namely, if (5.15) holds, then conclusion (1) of Theorem 4.2 fails for $\alpha = \dim A$.

We begin with an outline of our strategy. Let $0 < d < 1$ and consider the exceptional set $E \subset S^1$ for some (suitable) d-dimensional Borel set $A \subset \mathbb{R}^2$ with respect to the Euclidean projection $P^{\mathbb{E}}$. Then, by Theorem 3.1, E is a set of dimension $\leq d$. We construct the norm $\|\cdot\|$ such that the Gauss map for $\|\cdot\|$ blows up the exceptional set E to a set of positive $\mathscr{H}^1$-measure. This construction will roughly go as follows. Identify S^1 with the interval $[0, 2\pi)$. This identification will be denoted by $\alpha^{-1} : S^1 \to [0, 2\pi)$. We consider a suitable subset $K \subset \alpha^{-1}(E)$ and construct a strictly increasing function f that blows up the set K to a set of positive length. Then, the integral F of f will be strictly convex and C^1. Now, we roll the graph of F back up with α (resp. its extension h); see Figure 5.5. Thus, the image Γ of the graph of F will be a piece of the boundary of a strictly convex set which defines a norm $\|\cdot\|$ on $\mathbb{R}^2$, see Figure 5.8. We will show that the Gauss map of this norm restricted to Γ, will still behave like the function f in terms of its measure theoretic properties. Finally, we will apply arguments from the proof of Theorem 4.2 to conclude (5.15).

Now, we start with the formal proof. Let $0 < d < 1$. As established in [25], there exists a compact set $A \subset \mathbb{R}^2$ of dimension d such that $\dim(E) = d$ where

$$E := \{w \in S^1 : \dim(P^{\mathbb{E}}_{w^\perp}(A)) < d\}.$$

Moreover, E is a Borel set (see [24]) and $\mathscr{H}^1(E) = 0$ (see Theorem 3.1). Consider the parameterization $\alpha : [0, 2\pi) \to S^1$ given by $\alpha(t) := (\cos(t), \sin(t))$. Then, since α is locally bi-Lipschitz, it follows that $\dim(\alpha^{-1}(E)) = d$. Let $0 < s < d$. Then, by definition of the Hausdorff dimension, $\mathscr{H}^s(\alpha^{-1}(E)) = \infty$. Thus, by Theorem 8.13 in [30], there exists a compact set $K \subset \alpha^{-1}(E)$ with $0 < \mathscr{H}^s(K) < \infty$. We assume without loss of generality that $K \subset [0, 1]$. In particular, this yields

$$K \subset (\{t \in [0, 1] : \dim P^{\mathbb{E}}_{\alpha(t)^\perp}(A) < \dim A\}). \tag{5.16}$$

Now, define $f : [0, 1] \to [0, 1]$ by

$$f(t) := \frac{1}{2}\left(\frac{1}{\mathscr{H}^s(K)}\mathscr{H}^s([0, t] \cap K) + t\right). \tag{5.17}$$

We have seen this exact construction when K is the triadic Cantor set in Remark 5.19. Since K is compact, $[0, 1]\backslash K$ consists of countably many (relatively) open intervals in $[0, 1]$. Therefore, we can conclude by the same arguments as in Remark 5.19 that $f : [0, 1] \to [0, 1]$ is a strictly increasing homeomorphism and $\mathscr{L}^1(f(K)) = \frac{1}{2} > 0$. Next, we define the mapping $F : [0, 1] \to [0, 1]$ by

$$F(u) := \frac{1}{4}\int_0^u f(t)\mathrm{d}t.$$

Then, $F : [0, 1] \to [0, 1]$ is an injective and strictly convex C^1-mapping with $F(1) \le \frac{1}{4}$. Let $S := \{r \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} : t \in [0, 1],\, r \ge 0\} \subset \mathbb{R}^2$. Moreover, we define $h : [0, 1] \times [0, 1] \to S$ by $h(x, y) := (1 - y)\begin{pmatrix} \cos(x) \\ \sin(x) \end{pmatrix}$, and the curve $\gamma : [0, 1] \to S$ by $\gamma(t) := h(t, F(t))$. Thus, the curve γ parameterizes $h(\mathrm{Graph}(F))$, see Figure 5.5.

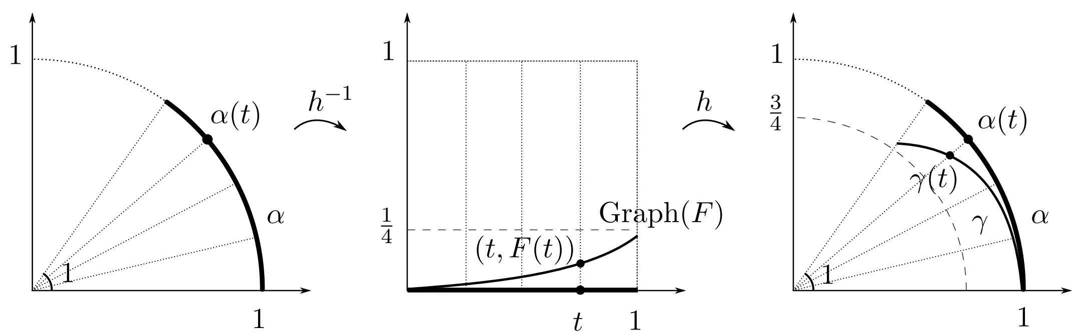

Figure 5.5. Construction of the curve γ from $F : [0, 1] \to [0, \frac{1}{4}]$.

Observe that for all $t \in [0,1]$,

$$\alpha(t) = \frac{\gamma(t)}{|\gamma(t)|}. \tag{5.18}$$

Moreover, γ is a regular C^1-curve and $\dot{\gamma}$ is given by

$$\begin{aligned}
\dot{\gamma}(t) &= \begin{pmatrix} -(1-F(t))\sin(t) & -\cos(t) \\ (1-F(t))\cos(t) & -\sin(t) \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{4}f(t) \end{pmatrix} && (5.19) \\
&= (1-F(t)) \begin{pmatrix} \cos(t+\frac{\pi}{2}) & -\sin(t+\frac{\pi}{2}) \\ \sin(t+\frac{\pi}{2}) & \cos(t+\frac{\pi}{2}) \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{4(1-F(t))}f(t) \end{pmatrix}. && (5.20)
\end{aligned}$$

Notice that since $0 \le F(t) \le \frac{1}{4}$, it follows that $4(1-F(t)) \ge 3$ and $\frac{1}{4(1-F(t))} \le \frac{1}{3}$.

Consider the curve $\beta : [0,1] \to S^1$, defined by $\beta(t) := \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|}$. We will now establish the following properties for β.

(i) $\beta : [0,1] \to S^1$ is an injective curve that travels in S^1 in counterclockwise direction from $\beta(0) = \binom{0}{1}$ to $\beta(1)$ where $\beta(1) = (\cos(s), \sin(s))$, with $s \in (\frac{\pi}{2}, \pi)$.

(ii) $\mathscr{H}^1(\beta(K)) > 0$.

Let us begin by defining shorter notations for the objects appearing in (5.20). For $t \in [0,1]$, we write

$$M(t) := \begin{pmatrix} \cos(t+\frac{\pi}{2}) & -\sin(t+\frac{\pi}{2}) \\ \sin(t+\frac{\pi}{2}) & \cos(t+\frac{\pi}{2}) \end{pmatrix}$$

and

$$v(t) = \begin{pmatrix} 1 \\ \frac{1}{4(1-F(t))}f(t) \end{pmatrix}.$$

Hence, $M(t) \in O(2)$, $v(t) \in (\{1\} \times [0, \frac{1}{3}]) \subset \mathbb{R}^2$ and $\dot{\gamma}(t) = (1-F(t))M(t)v(t)$, for all $t \in [0,1]$. Set $w(t) := \frac{v(t)}{|v(t)|}$ for $t \in [0,1]$. Then, by (5.20), and the fact that $M(t) \in O(2)$ for all $t \in [0,1]$, it follows that $\beta(t) = M(t)w(t)$.

Recall that the functions $f : [0,1] \to [0,1]$ as well as $F : [0,1] \to [0, \frac{1}{4}]$ are strictly increasing. Thus, in particular, $t \mapsto \frac{1}{4(1-F(t))}$ is strictly increasing. Also, recall that $\mathscr{H}^1(f(K)) > 0$. Hence, $\psi : [0,1] \to [0, \frac{1}{3}]$, defined by

$$\psi(t) := \frac{1}{4(1-F(t))}f(t)$$

also is strictly increasing, and, by Lemma 5.18, $\mathscr{H}^1(\psi(K)) > 0$.
Note that $\mathbb{R} \to (\{1\} \times \mathbb{R}) \subset \mathbb{R}^2, x \mapsto \binom{1}{x}$ is an isometric embedding (i.e. a 1-bi-Lipschitz mapping) and $v(t) = \binom{1}{\psi(t)}$. Therefore, $v : [0,1] \to \{1\} \times [0, \frac{1}{3}]$ is injective with $v(0) = \binom{1}{0}$ and $v(1) = \binom{1}{1/(4(1-F(1)))}$, and $\mathscr{H}^1(v(K)) > 0$.

Recall that $w(t) = \frac{v(t)}{|v(t)|}$, for $t \in [0,1]$. Thus, $w : [0,1] \to S^1$ is an injective curve that travels from $w(0) = v(0) = \binom{1}{0}$ to $w(1)$, see Figure 5.6.

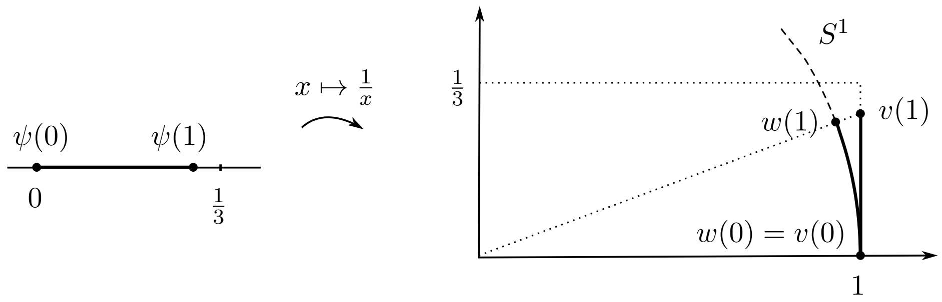

Figure 5.6. Construction of v and w from ψ.

For $t \in [0,1]$, denote by $\theta(t) \in [0, 2\pi)$ the counterclockwise angle from the x-axis to $w(t)$, thus

$$w(t) = \begin{pmatrix} \cos(\theta(t)) \\ \sin(\theta(t)) \end{pmatrix}. \tag{5.21}$$

Recall that $v(1) = \begin{pmatrix} 1 \\ 1/4(1-F(1))) \end{pmatrix}$ and notice that

$$\frac{1}{1/(4(1-F(1)))} = 4(1-F(1)) \geq 3 > \frac{\cos(\frac{\pi}{2}-1)}{\sin(\frac{\pi}{2}-1)}.$$

Therefore, it follows that $w(1) = \frac{v(1)}{|v(1)|} = \begin{pmatrix} \cos(\theta(1)) \\ \sin(\theta(1)) \end{pmatrix}$ with $\theta(1) \in (0, \frac{\pi}{2} - 1)$. Moreover, from the fact that $(\{1\} \times [0, \frac{1}{3}]) \to S^1$, $x \mapsto \frac{x}{|x|}$ is a bi-Lipschitz mapping, it follows that $\mathscr{H}^1(w(K)) > 0$.

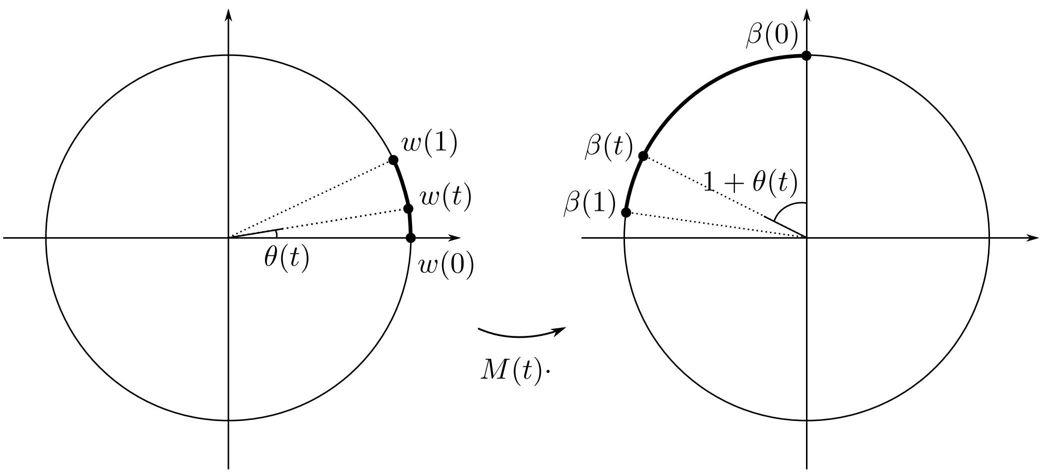

Figure 5.7. From $w(t)$ to $\beta(t)$ by left multiplication with $M(t)$, $t \in [0,1]$.

Now, consider the curve $\beta : [0,1] \to S^1,\ t \mapsto M(t)w(t)$. The matrix $M(t)$ is the matrix of the counterclockwise rotation about the angle $t + \frac{\pi}{2}$.

Thus, it follows that

$$\beta(t) = \begin{pmatrix} \cos(t + \frac{\pi}{2} + \theta(t)) \\ \sin(t + \frac{\pi}{2} + \theta(t)) \end{pmatrix}. \tag{5.22}$$

This makes $\beta : [0,1] \to S^1$ an injective curve that travels in S^1 in counterclockwise direction from $\beta(0) = \binom{0}{1}$ to $\beta(1) = \begin{pmatrix} \cos(s) \\ \sin(s) \end{pmatrix}$, where $s := 1 + \frac{\pi}{2} + \theta(1)$ and thus $s \in (1 + \frac{\pi}{2}, \pi)$. See Figure 5.7. This proves property (i). Moreover, it follows from (5.21) and (5.22) that $|\beta(t) - \beta(t')| \geq |w(t) - w(t')|$, for all $t, t' \in [0,1]$. Thus, by Lemma 5.17 and the fact that $\mathscr{H}^1(w(K)) > 0$, it follows that $\mathscr{H}^1(\beta(K)) > 0$. This proves property (ii).

Denote the image of $[0,1]$ under γ by Γ. From our bounds for the values of β at $t = 0$ and $t = 1$ (see property (i)), it follows that we can extend the union $\Gamma \cup (-\Gamma)$ to the image of a closed C^1-curve $\bar{\Gamma}$, by gluing arcs R and $-R$ to Γ and $-\Gamma$, such that the tangential directions at the gluing points agree, as illustrated in Figure 5.8.

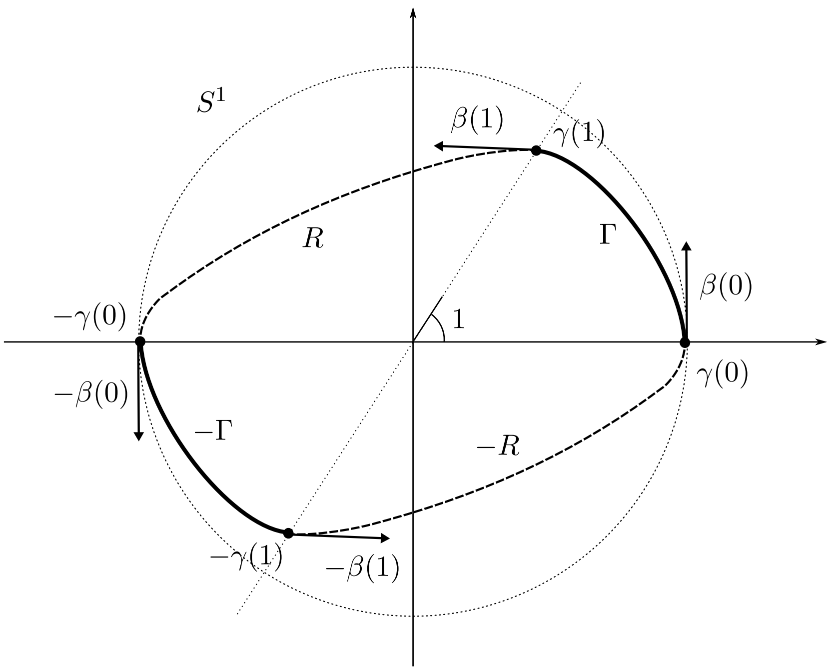

Figure 5.8. Normal sphere that contains the arc Γ.

Observe that by injectivity of β, see property (i), $\bar{\Gamma}$ is a simply closed curve that bounds a strictly convex, antipodally symmetric subset of $\mathbb{R}^2$ with non-empty interior. Hence, $\bar{\Gamma}$ defines a norm $\|\cdot\|$ on $\mathbb{R}^2$ by setting $S^1_{\|\cdot\|} := \bar{\Gamma}$. Moreover, since $\beta(t)$ is tangential to $\bar{\Gamma}$ at $\gamma(t) \in \bar{\Gamma}$ for $t \in [0,1]$, the Gauss map $G : S^1_{\|\cdot\|} \to S^1$ of the norm $\|\cdot\|$ in such points is given by

$$G(\gamma(t)) = R_{\frac{\pi}{2}}\beta(t), \tag{5.23}$$

where $R_{\frac{\pi}{2}}$ denotes the counterclockwise rotation about the angle $\frac{\pi}{2}$.

We will now prove that the family $P^{\|\cdot\|} : G(2,1) \times \mathbb{R}^2 \to \mathbb{R}^2$ of closest-point projections with respect to $\|\cdot\|$ satisfies (5.15). For this, first, recall from (4.4) and Lemma 5.6 that $\dim P^{\|\cdot\|}_{w^\perp}(A) = \dim P^{\mathbb{E}}_{\widetilde{\mathscr{G}}(w)^\perp}(A)$ where $\widetilde{\mathscr{G}}(w) = \frac{G^{-1}(w)}{|G^{-1}(w)|}$, for all $w \in S^1$. Thus, $\widetilde{\mathscr{G}}^{-1}(v) = G\big(\frac{v}{\|v\|}\big)$, for all $v \in S^1$. Using this, as well as the fact that norms in $\mathbb{R}^2$ are bi-Lipschitz equivalent (see (b) in Section 5.1), it follows that,

$$\begin{aligned}
&\mathscr{H}^1(\{w \in S^1 : \dim P^{\|\cdot\|}_{w^\perp}(A) < \dim A\}) \\
&= \mathscr{H}^1(\{u \in S^1 : \dim P^{\mathbb{E}}_{\widetilde{\mathscr{G}}(u)^\perp}(A) < \dim A\}) \\
&= \mathscr{H}^1(\{\widetilde{\mathscr{G}}^{-1}(u) : u \in S^1,\, \dim P^{\mathbb{E}}_{u^\perp}(A) < \dim A\}) \\
&= \mathscr{H}^1(\{G\big(\tfrac{u}{\|u\|}\big) : u \in S^1,\, \dim P^{\mathbb{E}}_{u^\perp}(A) < \dim A\}) \\
&= \mathscr{H}^1(\{G(v) : v \in S^1_{\|\cdot\|},\, \dim P^{\mathbb{E}}_{v^\perp}(A) < \dim A\}).
\end{aligned} \tag{5.24}$$

On the other hand, employing the fact that $\Gamma \subset S^1_{\|\cdot\|}$ as well as equation (5.23) yields

$$\begin{aligned}
&\mathscr{H}^1(\{G(v) : v \in S^1_{\|\cdot\|},\, \dim P^{\mathbb{E}}_{v^\perp}(A) < \dim A\}) \\
&\geq \mathscr{H}^1(\{G(v) : v \in \Gamma,\, \dim P^{\mathbb{E}}_{v^\perp}(A) < \dim A\}) \\
&= \mathscr{H}^1(\{G(\gamma(t)) : t \in [0,1],\, \dim P^{\mathbb{E}}_{v^\perp}(A) < \dim A\}) \\
&= \mathscr{H}^1(\{\beta(t) : t \in [0,1],\, \dim P^{\mathbb{E}}_{v^\perp}(A) < \dim A\}).
\end{aligned} \tag{5.25}$$

Moreover, by (5.16) and the fact that $\mathscr{H}^1(\beta(K)) > 0$, it follows that

$$\mathscr{H}^1(\{\beta(t) : t \in [0,1],\, \dim P^{\mathbb{E}}_{v^\perp}(A) < \dim A\}) \geq \mathscr{H}^1(\beta(K)) > 0. \tag{5.26}$$

Observe that (5.15) now follows from (5.24), (5.25) and (5.26). □

Notice that the Gauss map $G : S^1_{\|\cdot\|} \to S^1$ of the norm $\|\cdot\|$ constructed in the proof above may be a δ-Hölder mapping for some $\delta > 0$, depending on the geometry of K. This would imply that there exists a $C^{1,\delta}$-norm for which conclusions (1) and (2) of Theorem 4.2 fail. For example, if K was the triadic cantor set, the mapping $f : [0,1] \to [0,1]$ defined in (5.17) (and thus also the Gauss map G) would be $\frac{\log(2)}{\log(3)}$-Hölder. As pointed out in the introduction, the study of the geometry of the exceptional sets is an independent domain of research and so far, little is known about the geometry of the exceptional sets for the family of Euclidean projections (in the case $n = 2$ as well as in general). In particular, we do not know, whether a set like the triadic Cantor set appears as a subset of such exceptional sets.

Assume that we replaced the set A in the proof of Theorem 5.16 by a set $A \subset \mathbb{R}^2$ of dimension $d > 1$ whose exceptional set $E = \{w \in S^1 : \dim(P_{w^\perp}(A) < 1\}$ is a set of dimension d; the existence of such a set A is addressed in [13]. Then, it follows that there exists a norm $\|\cdot\|$ on $\mathbb{R}^2$, for which (2) of Theorem 4.2 fails.

In order to generalize the construction in the proof of Theorem 5.16 to families of projections $P : G(n, n-1) \times \mathbb{R}^n \to \mathbb{R}^n$ onto $(n-1)$-planes, one would have to find a suitable analog of the function f (see (5.17)) on an $(n-1)$-dimensional cube. We do not know how one could define such a function, given the fact that the structure of (compact subsets of) the exceptional sets of the family of Euclidean projections is unknown.
On the other hand, one could generalize Theorem 5.16 to families of projections onto lines by taking products and looking at the surface of revolution of $S^1_{\|\cdot\|}$ as the norm sphere in $\mathbb{R}^n$. However, such a result currently is not of great relevance since for most norms it is not known if the conclusions of Theorem 4.2 hold. This issue will be addressed in the subsequent section.

5.5 PROJECTIONS WITH CODIMENSION GREATER THAN ONE

In this section we address the case of projections onto m-planes induced by a norm for the cases when $m < n-1$. It turns out that our methods developed in the previous sections do in general not apply when $m < n-1$. Moreover, we will see that norms induced by an inner product represent an exception. Finally, we will outline that there exist many families of linear and surjective projections that are not induced by norms. This underlines the relevance of Theorem 4.2 independently of Theorem 5.5.

5.5.1 *Non-linearity for codimension greater than one*

As pointed out in Section 5.1, for every strictly convex norm $\|\cdot\|$ in $\mathbb{R}^n$ and for every $0 < m < n$, the family $P^{\|\cdot\|} : G(n, m) \times \mathbb{R}^n \to \mathbb{R}^n$ of closest-point projections with respect to $\|\cdot\|$ is well-defined. Nevertheless, our results in Sections 5.2 and 5.3 only cover the case when $\|\cdot\|$ is sufficiently regular and $m = n-1$. Both these restrictions are necessary for our methods of proof to work. In the case when $m = n-1$, our main tool for describing the projections $P_V : \mathbb{R}^n \to V$, $V \in G(n, m)$ is the Gauss map. The regularity of $\|\cdot\|$ in the first place guarantees the existence and regularity of the Gauss map. Once the Gauss map is known to be well-behaved it basically suffices to establish and exploit the fact that the projections P_V are linear maps, when $m = n-1$. However, in general projections $P : \mathbb{R}^n \to V$ onto m-planes $V \in G(n, m)$, with $m < n-1$, fail to be linear.

To see this consider the p-norm $\|\cdot\|_p$ on $\mathbb{R}^n$ for $2 \leq p < \infty$, defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \tag{5.27}$$

for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Then, $\|\cdot\|_2$ equals the Euclidean norm on $\mathbb{R}^n$.

Notice that $\|\cdot\|_p$ is k-times continuously differentiable in $\mathbb{R}^n\backslash\{0\}$ if and only if its p-th power $\|\cdot\|_p^p$ is k-times continuously differentiable in $\mathbb{R}^n\backslash\{0\}$. The map $t \mapsto |t|^p$ is k-times continuously differentiable in $\mathbb{R}$, whenever $p > k$. Moreover, the k-th differential at $t \in \mathbb{R}$ then equals $c(k,p)|t|^{p-k}$ and the constant $c(k,p)$ depends on k and p only. Hence, we can conclude that $\|\cdot\|_p$ is $C^{k,\delta}$ for some $\delta > 0$ whenever $k < p$. Then, since $\|\cdot\|_2$ is known to be C^∞, we may conclude that, for all $2 \leq p < \infty$, Theorem 5.5 applies and so does Theorem 5.9 for $L = K - 1$ and some $\delta > 0$.

Let $2 \leq p < \infty$ and by $P : G(n,m) \times \mathbb{R}^n \to \mathbb{R}^n$ denote the family of closest-point projections $P_V : \mathbb{R}^n \to V$, $V \in G(n,m)$ with respect to $\|\cdot\|_p$.

Proposition 5.20. *Let $2 \leq p < \infty$. Thus, $P : G(n,1) \times \mathbb{R}^n \to \mathbb{R}^n$ is a family of linear projections if and only if $p = 2$.*

Notice that it suffices to prove Proposition 5.20 for the case when $n = 3$. The proof is a straight-forward calculation.

Proof. By e_1, e_2, e_3 denote the standard basis of $\mathbb{R}^3$. Define $e = e_1 + e_2 + e_3$ and define $L \in G(3,1)$ by $L = \{te : t \in \mathbb{R}\}$. Then, for all $i = 1,2,3$, the projection of e_i onto L with respect to $\|\cdot\|_p$ is given by $P_L(e_i) = t_i e$ where $t = t_i$ minimizes $\|e_i - te\|_p$, or equivalently, $t = t_i$ minimizes $h_i(t) := \|e_i - te\|_p^p = |1-t|^p + 2|t|^p$.
Assume that $0 < t < 1$, then $h_i(t) = (1-t)^p + 2t^p$. Thus, setting $\dot{h}_i(t) = 0$, yields $-(1-t)^{p-1} + 2t^{p-1} = 0$, and hence, $t = (2^{\frac{1}{p-1}} + 1)^{-1}$. If we proceed in the same way, assuming $t \leq 0$ or $t \geq 1$, we arrive at a contradiction. Thus, since we know that $P_L(e_i)$ and thus a minimizing t_i exists, it follows that $t_i = (2^{\frac{1}{p-1}} + 1)^{-1}$ and

$$P_L(e_i) = (2^{\frac{1}{p-1}} + 1)^{-1} e,$$

for all $i = 1,2,3$. By an analogous argument, one can show that

$$P_L(e_i + e_j) = 2^{\frac{1}{p-1}} (2^{\frac{1}{p-1}} + 1)^{-1} e$$

for all $i \neq j$. Then, $P_L(e_i) + P_L(e_j) = P_L(e_i + e_j)$ if and only if $p = 2$. Hence, since the Euclidean projection $P_L^{\mathbb{E}} = P_L^2$ is known to be linear for all $L \in G(3,1)$, this completes the proof. $\square$

Notice that the p-norm can also be defined for $1 \leq p < 2$. A discussion of projections theorems for p-norms with $1 \leq p < 2$ in the case when $n = 2$ can be found in [3].

5.5.2 *Projections induced by an inner product*

We say that a norm $\|\cdot\|$ is induced by an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$, if $\|x\|^2 = \langle x, x \rangle$ for all $x \in \mathbb{R}^n$. It is a well known fact that a norm $\mathbb{R}^n$ is induced by some inner product if and only if $S_{\|\cdot\|}^{n-1}$ is the surface of an n-dimensional ellipsoid.

Recall that we denote the Euclidean inner product (the scalar product) in $\mathbb{R}^n$ by $\langle\cdot,\cdot\rangle$. Let $e_1,\dots,e_n$ be the standard basis of $\mathbb{R}^n$ which is an orthonormal basis with respect to $\langle\cdot,\cdot\rangle$. Moreover, let $\prec\cdot,\cdot\succ$ be an inner product on $\mathbb{R}^n$ and $\widetilde{e}_1,\dots,\widetilde{e}_n$ an orthonormal basis of $\mathbb{R}^n$ with respect to $\prec\cdot,\cdot\succ$. Then, the linear mapping $\Psi:(\mathbb{R}^n,\prec\cdot,\cdot\succ)\to(\mathbb{R}^n,\langle\cdot,\cdot\rangle)$ defined by $\Psi(\widetilde{e}_i)=e_i$ for all $i=1,\dots,n$, is an isometry in the sense that $\prec x,y\succ=\langle\Psi(x),\Psi(y)\rangle$ for all $x,y\in\mathbb{R}^n$. Hence, it follows that

$$P_V^{\|\cdot\|}(x)=\psi^{-1}\circ P_{\Psi(V)}^{\mathbb{E}}\circ\Psi(x), \tag{5.28}$$

for all $x\in\mathbb{R}^n$ and $V\in G(n,m)$. To see this, let $x\in\mathbb{R}^n$ and $V\in G(n,m)$, then by definition of $P^{\|\cdot\|}$, we have $\|x-P_V^{\|\cdot\|}(x)\|=\operatorname{dist}_{\|\cdot\|}(V,x)$. Since Ψ is an isometry, this implies that $|\Psi(x)-\Psi(P_V^{\|\cdot\|}(x))|=\operatorname{dist}_{\mathbb{E}}(\Psi(x),\Psi(V))$, and hence, by definition of the Euclidean projection, $P_{\Psi(V)}^{\mathbb{E}}(\Psi(x))=\Psi(P_V^{\|\cdot\|}(x))$ which implies (5.28).

Therefore, in particular, the projection $P_V^{\|\cdot\|}:\mathbb{R}^n\to V$ is linear and surjective for all $V\in G(n,m)$. Moreover, the mapping $\mathscr{G}$ associated with the family $P^{\|\cdot\|}:G(n,m)\times\mathbb{R}^n\to\mathbb{R}^n$ is given by Ψ. Since, Ψ is a linear bijection, $\mathscr{G}:G(n,m)\to G(n,m)$ is a smooth diffeomorphism of manifolds and thus preserves measure and dimension. Therefore, Theorem 5.5 applies and Definition 3.9 holds with $L=\infty$.

5.5.3 *Linear projections that are not induced by a norm*

In this section, we wish to point out that families of projections induced by norms represent a rather small part among all families of linear and surjective projection that satisfy the conditions of Theorem 4.2.

In the spirit of the methods from Section 5.2, every mapping $g:G(n,m)\to G(n,m)$ defines a family of linear and surjective projections $P:G(n,m)\times\mathbb{R}^n\to\mathbb{R}^n$ by setting $P_V(x)=P_{g(V)}^{\mathbb{E}}(x)$. The mapping $\mathscr{G}$ associated with this family of projections P as defined in (4.1), equals g. Thus, if g is dimension non-decreasing and has the inverse Lusin property for $\sigma_{n,m}$ (see Section 2.1), then Theorem 4.2 applies to the family $P:G(n,m)\times\mathbb{R}^n\to\mathbb{R}^n$.

In order for a mapping $\mathscr{G}:G(n,m)\to G(n,m)$ to be dimension non-decreasing and possessing the inverse Lusin property, properties such as continuity or injectivity are not required. However, for families of linear projections that are induced by a strictly convex C^1-norm it is known that $\mathscr{G}$ is given by the inverse Gauss map G^{-1}. Recall from Lemma 5.2 that G^{-1} is known to be a homeomorphism in this setting. Moreover, if $\mathscr{G}$ is given in terms of the inverse Gauss map of a strictly convex C^1-norm, by conclusion (ii) of Lemma 5.2, $\mathscr{G}$ possesses at least two fixed points.

This allows the construction of many families of linear and surjective projections that are not induced by a norm and for which Theorem 4.2 holds. In particular, it is easy to explicitly define and illustrate such examples in $\mathbb{R}^2$.

For all angles $\theta \in [0, 2\pi)$, let $v_\theta = \binom{\cos\theta}{\sin\theta} \in S^1$ and $L_\theta = v_\theta^\perp$. Consider a mapping $\alpha : [0, 2\pi) \to (0, \pi)$ for which

$$\alpha(\theta) = \alpha(\theta + \pi) \tag{5.29}$$

for all $\theta \in [0, \pi)$. We define $w_\theta \in S^1$ to be

$$w_\theta := \begin{pmatrix} \cos(\theta - \frac{\pi}{2} + \alpha(\theta)) \\ \sin(\theta - \frac{\pi}{2} + \alpha(\theta)) \end{pmatrix},$$

see left-hand side of Figure 5.9. Define a family of projections $P^\alpha : G(2, 1) \times \mathbb{R}^2 \to \mathbb{R}^2$ as follows. For $\theta \in [0, \pi)$ and $x \in \mathbb{R}^2$, let $P^\alpha_{L_\theta}(x)$ be the intersection point of the line $L_\theta = v_\theta^\perp$ with the affine line $\{x + rw_\theta : r \in \mathbb{R}\}$; see right-hand side of Figure 5.9.

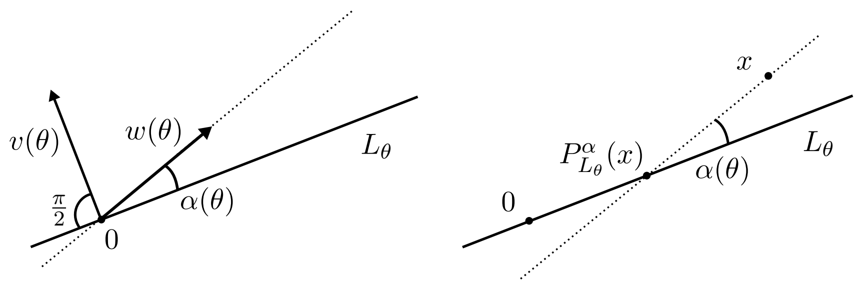

Figure 5.9. Construction of the projection $P^\alpha_{L_\theta}$.

Then, for all L_θ, $\theta \in [0, 2\pi)$, we obtain $\operatorname{Ker} P^\alpha_{L_\theta} = \{x + rw_\theta : r \in \mathbb{R}\}$. Thus, the mapping $\mathscr{G} : G(2, 1) \to G(2, 1)$ for the family P^α is given by $\mathscr{G}(v_\theta^\perp) = w_\theta^\perp$. Thus, by identification of $G(2, 1)$ with S^1, and S^1 with $[0, 2\pi)$, the mapping $\mathscr{G}$ can be viewed as the mapping $\widetilde{\mathscr{G}} : [0, 2\pi) \to [0, 2\pi)$ given by

$$\widetilde{\mathscr{G}}(\theta) = \theta + \alpha(\theta),$$

where angles $\widetilde{\mathscr{G}}(\theta)$ that are greater than 2π are identified with $\widetilde{\mathscr{G}}(\theta) - 2\pi$. Assume that there exists a strictly convex C^1-norm $\|\cdot\|$ on $\mathbb{R}^2$ such that the family of projections induced by $\|\cdot\|$ equals the family P^α. By the considerations above, $\widetilde{\mathscr{G}}$ is a homeomorphism with at least four fixed points (where always two and two correspond to antipodal directions in S^1). Thus, every mapping $\alpha : [0, 2\pi) \to (0, \pi)$ such that $\theta \mapsto \theta + \alpha(\theta)$ is dimension non-decreasing and has the inverse Lusin property, but is not a homeomorphism with at least four fixed points, yields a family of linear and surjective projections that is not induced by a norm and satisfies Theorem 4.2. For example, consider the case when $\alpha : [0, 2\pi) \to (0, \pi)$ is constant, i.e., $\alpha(\theta) = c$, for all $\theta \in [0, 2\pi)$, where $c \in (0, \pi)$ a constant. Then, P^α is induced by a strictly convex C^1-norm if and only if $c = 0$. Moreover, in this case the norm is the Euclidean norm. However, for any choice of $c \in (0, \pi)$, Theorem 4.2 applies to P^α.

Chapter 6

RIEMANNIAN MANIFOLDS OF CONSTANT SECTIONAL CURVATURE

6.1 HYPERBOLIC PLANE AND TWO-SPHERE

In this section, we establish Marstrand-type projection theorems for the family of orthogonal projections in the hyperbolic 2-plane as well as in an open half-sphere of S^2. For this, we will prove that a slightly adapted version of the respective family of projections satisfies differentiable transversality in the sense of Definition 3.5. Our proofs are based on standard tools from hyperbolic and spherical trigonometry that can be found in [6], [8], and [10].

The content of this section was published in [4].

6.1.1 *Hyperbolic plane*

By $\mathbb{H}^2$ denote the hyperbolic 2-plane and by d the hyperbolic metric on $\mathbb{H}^2$. We fix a base point $p \in \mathbb{H}^2$ and identify the tangent plane $T_p\mathbb{H}^2$ with $\mathbb{R}^2$ and consider the exponential mapping $\exp_p : \mathbb{R}^2 \to \mathbb{H}^2$ of p for $\mathbb{H}^2$. Let $L \in G(2,1)$. Then $\exp_p(L)$ is a geodesic line in $\mathbb{H}^2$ and thus a geodesically convex subspace of $\mathbb{H}^2$. Since $\mathbb{H}^2$ is simply connected and of non-positive sectional curvature, it follows that for all $x \in \mathbb{H}^2$, there exists a unique point $y \in \exp_p(L)$, such that $\operatorname{dist}(x, \exp_p(L)) = d(x,y)$. We call this point $y \in \exp_p(L)$ the projection of x to $\exp_p(L)$ and denote it by $P_L(x)$. We will therefore call the mapping

$$P : G(2,1) \times \mathbb{H}^2 \to \mathbb{H}^2, \tag{6.1}$$

defined by $P(L,x) := P_L(x)$ for $x \in \mathbb{H}^2$ and $L \in G(2,1)$, the family of closest-point projections in $\mathbb{H}^2$. Moreover, Proposition 2.4 in [8] implies that the mappings $P_L : \mathbb{H}^2 \to \exp_p(L)$ are 1-Lipschitz and that for all $x \in \mathbb{H}^2$ and $L \in G(2,1)$ the geodesic segment $[x, P_L(x)]$ intersects $\exp_p(L)$ orthogonally in the point $P_L(x)$. Therefore, we will sometimes also refer to $P : G(2,1) \times \mathbb{H}^2 \to \mathbb{H}^2$ as the family of orthogonal projections (along geodesics) in $\mathbb{H}^2$. In particular, it follows that for all $A \subseteq \mathbb{H}^2$, $\dim P_L(A) \leq \dim A$.

In order to establish differentiable transversality for the family of orthogonal projections in $\mathbb{H}^2$, we define a family of abstract projections associated with $P : G(2,1) \times \mathbb{H}^2 \to \mathbb{H}^2$ as follows.

For $\theta \in \mathbb{R}$, define $v_\theta := \binom{\cos\theta}{\sin\theta} \in S^1$ and $L_\theta \in G(2,1)$ to be the line

$$L_\theta := \{r\, v_\theta : r \in \mathbb{R}\}. \tag{6.2}$$

Define the family of abstract projections $\Pi : \mathbb{R} \times \mathbb{H}^2 \to \mathbb{R}$ by

$$\Pi(\theta, x) := \pm d(p, P_{L_\theta}(x)), \tag{6.3}$$

where $\pm$ is to be interpreted as follows: $\Pi(\theta, x) = d(p, P_{L_\theta}(x))$, if $P_{L_\theta}(x) = rv_\theta$ for $r \geq 0$, and $\Pi(\theta, x) = -d(p, P_{L_\theta}(x))$, if $P_{L_\theta}(x) = rv_\theta$ for $r < 0$. Notice that from this definition, it immediately follows that $\Pi : \mathbb{R} \times \mathbb{H}^2 \to \mathbb{R}$ is continuous and that for all $x, y \in \mathbb{H}^2$, $\theta \in \mathbb{R}$,

$$d(P_{L_\theta}(x), P_{L_\theta}(y)) = |\Pi(\theta, x) - \Pi(\theta, y)|, \tag{6.4}$$

and

$$\Pi(\theta + \pi, x) = \Pi(\theta, x). \tag{6.5}$$

Hence, by (6.4) and the fact that the projections $P_L : \mathbb{H}^2 \to \exp_p(L)$ are 1-Lipschitz, it follows that the abstract projections $\Pi_\theta : \mathbb{H}^2 \to \mathbb{R}$, given by $\pi_\theta(x) := \Pi(\theta, x)$, are 1-Lipschitz, and thus dimension non-increasing.

In order to express Π_θ in a way that allows us to study its transversality and regularity properties, we will employ some basic facts from hyperbolic trigonometry.
Consider a geodesic triangle in $\mathbb{H}^2$ with side lengths a, b, c and opposite angles α, β, γ. It holds that

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos\alpha. \tag{6.6}$$

This formula is called the hyperbolic law of cosines. A proof can be found in [8]. Now, consider a geodesic triangle as with $\gamma = \frac{\pi}{2}$. From (6.6), we obtain $\cosh c = \cosh b \cosh a$ and $\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos\alpha$. Thus, $\frac{\cosh c}{\cosh b} = \cosh b \cosh c - \sinh b \sinh c \cos\alpha$ which implies $-\frac{\cosh c}{\cosh b} \sinh^2 b = -\sinh b \sinh c \cos\alpha$. In consequence, for geodesic triangles with $\gamma = \frac{\pi}{2}$:

$$\tanh b = \tanh c \cos\alpha. \tag{6.7}$$

Now, for each point $x \in \mathbb{H}$ and every angle $\theta \in \mathbb{R}$, denote by $\alpha_{x,\theta} \in [0, 2\pi)$ the counterclockwise angle from v_θ to the geodesic segment connecting the base point p to x. Let $\theta \in \mathbb{R}$ and $x \in \mathbb{H}^2$ such that $0 \leq \alpha_{x,\theta} < \frac{\pi}{2}$. Then, $P_{L_\theta}(x) = rv_\theta$ where $r = d(P_{L_\theta}(x), p) > 0$ and the three points p, x and $P_{L_\theta}(x)$ span a geodesic triangle with side lengths $a = d(x, P_{L_\theta}(x))$, $b = d(p, P_{L_\theta}(x))$, $c = d(p, x)$ and opposite angles $\alpha = \alpha_{x,\theta}$, β, $\gamma = \frac{\pi}{2}$. By (6.7), it follows that $\tanh d(p, P_{L_\theta}(x)) = \tanh d(p, x) \cos(\alpha_{x,\theta})$. Hence, by the definition of Π_θ and the fact that $P_{L_\theta}(x) = rv_\theta$, with $r = d(P_{L_\theta}(x), p) > 0$, it follows that $\tanh \Pi(\theta, x) = \tanh d(p, x) \cos(\alpha_{x,\theta})$, for all $\theta \in \mathbb{R}$ and all $x \in \mathbb{H}^2$. The other cases can be treated similarly. Hence, for all $\theta \in \mathbb{R}$ and all $x \in \mathbb{H}^2$,

$$\tanh \Pi(\theta, x) = \tanh d(p, x) \cos(\alpha_{x,\theta}). \tag{6.8}$$

For each point $x \in \mathbb{H}^2$, let $\alpha(x) \in \mathbb{R}$, denote the counterclockwise angle from v_0 to the geodesic segment connecting the base point p to x, by $\alpha(x) \in [0, 2\pi)$. It is easy to check that $\cos(\alpha_{x,\theta}) = \cos(\theta - \alpha(x))$ for all $\theta \in (0, \pi)$. In conclusion:

$$\tanh d(p, P_{L_\theta}(x)) = \tanh d(p, x) \cos(\theta - \alpha(x)), \tag{6.9}$$

for all $x \in \mathbb{H}^2$ and $\theta \in \mathbb{R}$. Motivated by (6.8), we introduce a new family of abstract projections $\widetilde{\Pi} : \mathbb{R} \times \mathbb{H}^2 \to \mathbb{R}$ by

$$\widetilde{\Pi}(\theta, x) := \tanh d(p, x) \cos(\alpha(x) - \theta). \tag{6.10}$$

Note that thus, for all $\theta \in \mathbb{R}$ and $x \in \mathbb{H}^2$,

$$\widetilde{\Pi}(\theta, x) = \tanh(\Pi(\theta, x). \tag{6.11}$$

Therefore, $\widetilde{\Pi} : \mathbb{R} \times \mathbb{H}^2 \to \mathbb{R}$ is continuous with respect to d. Moreover, note that tanh is a 1-Lipschitz function on $\mathbb{R}$ and recall that for all $\theta \in \mathbb{R}$, Π_θ is 1-Lipschitz. Therefore, $\widetilde{\Pi}_\theta : \mathbb{H}^2 \to \mathbb{R}$ is 1-Lipschitz for all $\theta \in \mathbb{R}$.

Let Ω be a closed ball with center p and a large radius $R > 0$ in $\mathbb{H}^2$ and consider the restricted family of projections $\widetilde{\Pi} : \mathbb{R} \times \Omega \to \mathbb{R}$. We will now prove the following main result of this section.

Theorem 6.1. *The family of abstract projections $\widetilde{\Pi} : \mathbb{R} \times \Omega \to \mathbb{R}$ satisfies differentiable transversality with $L = \infty$.*

Since tanh is locally bi-Lipschitz on $\mathbb{R}$, as a consequence of Theorem 3.7, Theorem 6.1 and (6.10), the conclusions of Theorem 3.7 as well as Theorem 3.14 hold for the family $\Pi : \mathbb{R} \times \Omega \to \mathbb{R}$ of abstract orthogonal projections on the hyperbolic plane with parameter $L = \infty$. This can be formulated equivalently for the family $P : G(2,1) \times \mathbb{H}^2 \to \mathbb{H}^2$, defined in (6.1), as follows.

Corollary 6.2. *For all Borel sets $A \subseteq \mathbb{H}^2$, the following hold.*

(1) *If $\dim A \leq 1$, then*
  - *(a)* $\dim(P_L A) = \dim A$ *for $\sigma_{2,1}$-a.e. $L \in G(2,1)$,*
  - *(b) For $0 < \alpha \leq \dim A$, $\dim(\{L \in G(2,1) : \dim(P_L A) < \alpha\}) \leq \alpha$.*

(2) *If $\dim A > 1$, then*
  - *(a)* $\mathscr{H}^1(P_L A) > 0$ *for $\sigma_{2,1}$-a.e. $L \in G(2,1)$,*
  - *(b)* $\dim(\{L \in G(n,m) : \mathscr{H}^1(P_L A) = 0\}) \leq 2 - \dim A$.

Moreover, a set $\widetilde{A} \subseteq \mathbb{H}^2$ with $\mathscr{H}^1(\widetilde{A}) < \infty$ is purely 1-unrectifiable if and only if $\mathscr{H}^1(P_L(\widetilde{A})) = 0$ for $\sigma_{2,1}$-a.e. $L \in G(2,1)$.

Consider the mapping $\widetilde{\Phi} : \mathbb{R} \times ((\Omega \times \Omega)\backslash\text{Diag})$, $(\theta, x, y) \to \widetilde{\Phi}(\theta, x, y)$, given by

$$\widetilde{\Phi}(\theta, x, y) = \frac{\widetilde{\Pi}(\theta, x) - \widetilde{\Pi}(\theta, y)}{d(x, y)}, \tag{6.12}$$

where Diag denotes the diagonal of $\Omega \times \Omega$. The following lemma will be crucial for the proof of Theorem 6.1.

Lemma 6.3. *There exists a mapping $D : (\Omega \times \Omega)\backslash\text{Diag} \to [0, \infty)$ and a mapping $\hat{\theta} : (\Omega \times \Omega)\backslash\text{Diag} \to [0, 2\pi)$ such that*

(1) *for all $(x, y) \in (\Omega \times \Omega)\backslash\text{Diag}$ and all angles $\theta \in \mathbb{R}$,*

$$\widetilde{\Pi}(\theta, x) - \widetilde{\Pi}(\theta, y) = D(x, y) \cos(\theta - \hat{\theta}(x, y)),$$

(2) *there exist constants $c > 0$ and $C > 0$, such that for all $(x, y) \in (\Omega \times \Omega)\backslash\text{Diag}$,*

$$c \leq \frac{D(x, y)}{d(x, y)} \leq C.$$

Proof. Let $(x, y) \in (\Omega \times \Omega)\backslash\text{Diag}$. Throughout this proof, we will use the following notation.

$$\begin{aligned} &d_1 = d(p, x),\ d_2 = d(p, y),\ d = d(x, y), \\ &\widetilde{d_1} = \tanh d(x, p),\ \widetilde{d_2} = \tanh d(y, p). \end{aligned} \tag{6.13}$$

By (6.9), we can thus write $\widetilde{\Pi}(\theta, x) = \widetilde{d_1} \cos(\theta - \alpha(x))$ and $\widetilde{\Pi}(\theta, y) = \widetilde{d_2} \cos(\theta - \alpha(y))$, for all $\theta \in \mathbb{R}$. In order to make the calculations clearer, write $\alpha = \theta - \alpha(y)$ and $\alpha_0 = \alpha(x) - \alpha(y)$. Thus, we obtain

$$\begin{aligned} \widetilde{\Pi}(\theta, x) &= \widetilde{d_1} \cos(\alpha - \alpha_0), \\ \widetilde{\Pi}(\theta, y) &= \widetilde{d_2} \cos(\alpha), \end{aligned} \tag{6.14}$$

and by an elementary calculation,

$$\widetilde{\Pi}(\theta, x) - \widetilde{\Pi}(\theta, y) = (\widetilde{d_1} \cos \alpha_0 - \widetilde{d_2}) \cos \alpha + \widetilde{d_1} \sin \alpha_0 \sin \alpha. \tag{6.15}$$

Define

$$\begin{aligned} A &= \widetilde{d_1} \cos \alpha_0 - \widetilde{d_2}, \\ B &= \widetilde{d_1} \sin \alpha_0. \end{aligned} \tag{6.16}$$

Thus, in particular, A and B cannot both be 0, since $(x, y) \notin \text{Diag}$. This allows us to make the following definition.

Let $\hat{\alpha} \in (0, 2\pi)$ be the angle that satisfies

$$\cos\hat{\alpha} = \frac{A}{\sqrt{A^2+B^2}} \quad \text{and} \quad \sin\hat{\alpha} = \frac{B}{\sqrt{A^2+B^2}}. \tag{6.17}$$

From (6.15) it follows that $\widetilde{\Pi}(\theta, x) - \widetilde{\Pi}(\theta, y) = \sqrt{A^2+B^2}\cos(\alpha - \hat{\alpha})$. Set $\hat{\theta} := \alpha(y) + \hat{\alpha}$ and $D := \sqrt{A^2+B^2}$. Observe that by their definition, both D and $\hat{\theta}$ are independent of θ. Thus, $D = D(x, y)$ and $\hat{\theta} = \hat{\theta}(x, y)$ are well-defined functions on $(\Omega \times \Omega)\backslash\mathrm{Diag}$. Moreover, by definition of $\alpha, \hat{\alpha}$ and $\hat{\theta}$, we conclude

$$\widetilde{\Pi}(\theta, x) - \widetilde{\Pi}(\theta, y) = D\cos(\theta - \hat{\theta}).$$

This completes the proof of Claim (1) in Lemma 6.3.
In order to prove, it suffices to show that $c \leq \frac{D(x,y)}{d(x,y)} \leq C$ for constants $c > 0$ and $C > 0$ independent of x and y.

By the hyperbolic law of cosines (6.6), applied to the geodesic triangle spanned by p, x and y, it holds that $\cosh d = \cosh d_1 \cosh d_2 - \sinh d_1 \sinh d_2 \cos\alpha_0$. This implies

$$-2\tanh d_1 \tanh d_2 \cos\alpha_0 = 2\left(\frac{\cosh d}{\cosh d_1 \cosh d_2} - 1\right). \tag{6.18}$$

Applying (6.16) and (6.18), as well as elementary product-to-sum identities for hyperbolic and trigonometric functions, yields

$$A^2 + B^2 = \frac{2\cosh d \cosh d_1 \cosh d_2 - \cosh^2 d_1 - \cosh^2 d_2}{\cosh^2 d_1 \cosh^2 d_2}. \tag{6.19}$$

Note that the product $\cosh d_1 \cosh d_2$ is greater than 1 and is bounded from above since $x, y \in \Omega$ and Ω is compact. So we can derive the following upper bound for $A^2 + B^2$:

$$A^2 + B^2 \leq \left(\frac{1}{\cosh^2 d_1} + \frac{1}{\cosh^2 d_1}\right)(\cosh d - 1) \leq 2(\cosh d - 1).$$

Hence, we conclude that

$$\frac{\sqrt{A^2+B^2}}{d} \leq \sqrt{2}\frac{\sqrt{\cosh d - 1}}{d}.$$

Note that $d \mapsto \frac{\sqrt{\cosh d - 1}}{d}$ is a continuous function in $d > 0$ and that the limit towards 0 is

$$\lim_{d \to 0^+} \frac{\sqrt{\cosh d - 1}}{d} = \frac{1}{\sqrt{2}} < \infty.$$

Thus, by the compactness of Ω, it follows that

$$\frac{\sqrt{A^2+B^2}}{d} \leq C$$

for some constant $C > 0$ only depending on the diameter of Ω. This proves the right-hand inequality in (2). Now, let us prove the left-hand inequality.

Using the notation from (6.13), we define $\rho = d_1 - d_2$. By the triangle inequality $\rho \in [-d, d]$, i.e., $|d| \geq |\rho|$ and therefore $\cosh d \geq \cosh \rho$. The following calculation only uses the definition of ρ and elementary calculation rules for cosh.

$$\begin{aligned}
&2\cosh d \cosh d_1 \cosh d_2 - \cosh^2 d_1 - \cosh^2 d_2 \\
= \ & 2\cosh d \cosh(d_2+\rho)\cosh d_2 - \cosh^2(d_2+\rho) - \cosh^2 d_2 \\
= \ & \cosh d(\cosh(2d_2+\rho)+\cosh\rho) - \frac{1}{2}(\cosh(2(d_2+\rho))+1) - \frac{1}{2}(\cosh(2d_2)+1) \\
= \ & \cosh d(\cosh(2d_2+\rho)+\cosh\rho) - \frac{1}{2}(\cosh(2(d_2+\rho)+\cosh(2d_2)) - 1 \\
= \ & \cosh d(\cosh(2d_2+\rho)+\cosh\rho) - \cosh(2d_2+\rho)\cosh\rho - 1 \\
= \ & \cosh d\cosh\rho - 1 + (\cosh d - \cosh\rho)\cosh(2d_2+\rho) \\
\geq \ & \cosh d\cosh\rho - 1 \geq \cosh d - 1.
\end{aligned}$$

Note that from the Taylor series representation of cosh, it follows that $\cosh d - 1 \geq \frac{1}{2}d^2$. Thus, the estimate,

$$2\cosh d\cosh d_1\cosh d_2 - \cosh^2 d_1 - \cosh^2 d_2 \geq \frac{1}{2}d^2, \tag{6.20}$$

follows. Now, since $x, y \in \Omega$ and Ω compact, there exists a constant $\widetilde{c} > 0$ (only depending on Ω) such that $\frac{1}{\cosh^2 d_1 \cosh^2 d_2} \geq \widetilde{c}$. Consequently, by (6.19) and (6.20), it follows that $\frac{\sqrt{A^2+B^2}}{d} \geq c$ for $c = \sqrt{\frac{\widetilde{c}}{2}}$ which completes the proof. □

Proof of Theorem 6.1. By (6.10), it follows that $\theta \mapsto \widetilde{\Pi}(\theta, x)$ is C^∞ and that the mappings $(\theta, x) \mapsto \frac{\mathrm{d}^l}{\mathrm{d}\theta^l}\theta \mapsto \widetilde{\Pi}(\theta, x)$, for all $l \in \mathbb{N}$, are continuous. Then, since Ω and S^1 are compact, the first condition in (a) in Definition 3.5 is satisfied (for $L = \infty$). By (ii) in Remark 3.6 we may neglect the second condition in (a).

From Lemma 6.3, it follows that

$$\frac{\mathrm{d}^l}{\mathrm{d}\theta^l}\Phi(\theta, x, y) \in \left\{ \pm\frac{D(x,y)}{d(x,y)}\sin(\theta - \hat{\theta}(x,y)),\ \pm\frac{D(x,y)}{d(x,y)}\cos(\theta - \hat{\theta}(x,y)) \right\} \tag{6.21}$$

for all $x, y \in (\Omega \times \Omega)\backslash\mathrm{Diag}$, $\theta \in \mathbb{R}$ and $l \in \mathbb{N} \cup 0$. Thus, as in the proof of (a) above, the family $\widetilde{\Pi} : \mathbb{R} \times \Omega \to \mathbb{R}$ satisfies (c) in Definition 3.5 with $L = \infty$ and $\delta = 0$.

Now, let $c' > 0$ such that $c' < \frac{c}{10}$ for the constant c from Lemma 6.3. Assume that

$|\Phi(\theta,x,y)| \leq c'$. Applying Lemma 6.3, yields

$$|\cos(\theta-\hat{\theta}(x,y))| \leq c'\frac{d(x,y)}{D(x,y)} \leq \frac{c'}{c} < \frac{1}{10}$$

and hence, $|\sin(\theta-\hat{\theta}(x,y))| \geq \frac{1}{10}$. By Lemma 6.3,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\Phi(\theta,x,y) = -\frac{D(x,y)}{d(x,y)}\sin(\theta-\hat{\theta}(x,y)),$$

and thus it follows that $\left|\frac{\mathrm{d}}{\mathrm{d}\theta}\Phi(\theta,x,y)\right| \geq \frac{c}{10}$. Hence, (b) from Definition 3.5 is satisfied as well. □

6.1.2 *Two-sphere*

Consider the Euclidean two-sphere S^2 embedded in $\mathbb{R}^2$, equipped with the angular metric d. Fix a base point $p \in S^2$. Identify the tangent plane T_pS^2 with $\mathbb{R}^2$ and consider the exponential mapping $\exp_p : \mathbb{R}^2 \to S^2$. Let $L \in G(2,1)$, then $\exp_p(L)$ is a (simply closed) geodesic line in S^2. Let $\Omega \subset S^2$ be the closed ball in S^2 with radius $0 < \mathbb{R} < \frac{\pi}{2}$ and center p.

Observe that due to the restriction $R < \frac{\pi}{2}$ for the radius of Ω the orthogonal projection of Ω onto each geodesic line through p is well defined. Namely, for all $x \in \Omega$ and $L \in G(2,1)$, there exists a unique point $q \in \exp_p(L)$, such that $d(x,q) = \mathrm{dist}(x,\exp_p(L))$; see [8], pages 176–178. Denote q by $P_L(x)$. Moreover, by the same argument as in the hyperbolic plane, the geodesic segment connecting x to $P_L(x)$ is orthogonal to $\exp_p(L)$. Therefore, we call the mapping $P : G(2,1) \times \Omega \to \Omega$ defined by $P(L,x) = P_L(x)$, for all $x \in \Omega$ and $L \in G(2,1)$ the family of orthogonal projections. In contrast to the previous section (hyperbolic plane), P_L is not 1-Lipschitz. However, for all $L \in G(2,1)$, P_L is M-Lipschitz for some constant $M > 0$ that only depends on R, and moreover, $P_L(x) \in \Omega$ for all $L \in G(2,1)$ and $x \in \Omega$. In particular, it follows that for all $A \subseteq \Omega$, $\dim P_L(A) \leq \dim A$.

Let $L_\theta \in G(2,1)$ for $\theta \in \mathbb{R}$ as in (6.2) and define the family of abstract projection $\Pi : \mathbb{R} \times \Omega \to \mathbb{R}$ by

$$\Pi(\theta,x) := \pm \mathrm{d}(p, P_{L_\theta}x). \tag{6.22}$$

where the notation $\pm$ is interpreted as in (6.3). It is immediate from this definition that

$$\mathrm{d}(P_\theta x, P_\theta y) = |\Pi(\theta,x) - \Pi(\theta,y)|. \tag{6.23}$$

The following formula is called the spherical law of cosines, a proof can be found in [8]. For a geodesic triangle with side lengths a,b,c, each $< \pi$, and opposite angles α,β,γ, it holds that

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha. \tag{6.24}$$

Applying the spherical law of cosines twice, yields

$$\tan b = \tan c \cos \alpha, \tag{6.25}$$

where $\gamma = \frac{\pi}{2}$. The proof of (6.25) is analogous to the proof of (6.7).
For each point $x \in \Omega$ and every angle $\theta \in \mathbb{R}$, let us denote by $\alpha_{x,\theta} \in [0, 2\pi)$ the counterclockwise angle from v_θ to the geodesic segment connecting the base point p to x. Moreover, denote the counterclockwise angle from v_0 to the geodesic segment connecting the base point p to x, by $\alpha(x) \in [0, 2\pi)$. An argument similar to the proof of (6.8) and (6.9) yields that

$$\begin{aligned} \tan \Pi_\theta x &= \tan(\mathrm{d}(p,x)) \cos(\alpha_{x,\theta}) \\ &= \tan(\mathrm{d}(p,x)) \cos(\theta - \alpha(x)). \end{aligned} \tag{6.26}$$

Motivated by (6.26), we define a new family of abstract projections $\widetilde{\Pi} : \mathbb{R} \times \Omega \to \mathbb{R}$, by

$$\widetilde{\Pi}(\theta, x) := \tan(\mathrm{d}(p,x)) \cos(\theta - \alpha(x)). \tag{6.27}$$

Then, for all $\theta \in \mathbb{R}$ and $x \in \Omega$, we obtain

$$\widetilde{\Pi}(\theta, x) = \tan(\Pi(\theta, x)) \,. \tag{6.28}$$

Thus, $\widetilde{\Pi}$ is continuous with respect to the metric d, and for all $\theta \in \mathbb{R}$, $\widetilde{\Pi}_\theta$ is Lipschitz for some Lipschitz constant that only depends on the radius R of Ω.
Now, for all angles $\theta \in \mathbb{R}$ and all pairs of distinct points $x, y \in \Omega$ define,

$$\Phi(\theta, x, y) = \frac{\widetilde{\Pi}(\theta, x) - \widetilde{\Pi}(\theta, y)}{\mathrm{d}(x,y)}.$$

We will now prove the following main result of this section.

Theorem 6.4. *The family of abstract projections $\widetilde{\Pi} : \mathbb{R} \times \Omega \to \mathbb{R}$ satisfies differentiable transversality with $L = \infty$.*

Since tan is bi-Lipschitz on $[-R, R]$ where $0 < R < \frac{\pi}{2}$ is the radius of Ω, the following corollary is a straight-forward consequence of Theorem 3.7 and Theorem 6.4.

Corollary 6.5. *Corollary 6.2 (with $\mathbb{H}^2$ replaced by Ω) holds for the family of orthogonal projections in the half-sphere $P : G(2,1) \times \Omega \to \Omega$.*

Consider the mapping $\widetilde{\Phi} : \mathbb{R} \times ((\Omega \times \Omega)\backslash \mathrm{Diag})$, $(\theta, x, y) \to \widetilde{\Phi}(\theta, x, y)$, given by

$$\widetilde{\Phi}(\theta, x, y) = \frac{\widetilde{\Pi}(\theta, x) - \widetilde{\Pi}(\theta, y)}{d(x,y)}, \tag{6.29}$$

where Diag denotes the diagonal of $\Omega \times \Omega$. The following lemma will be crucial for the proof of Theorem 6.4.

Lemma 6.6. *There exists a mapping $D : (\Omega \times \Omega)\backslash\mathrm{Diag} \to [0, \infty)$ and a mapping $\hat{\theta} : (\Omega \times \Omega)\backslash\mathrm{Diag} \to [0, 2\pi)$ such that*

(1) *for all $(x, y) \in (\Omega \times \Omega)\backslash\mathrm{Diag}$ and all angles $\theta \in \mathbb{R}$,*

$$\widetilde{\Pi}(\theta, x) - \widetilde{\Pi}(\theta, y) = D(x, y)\cos(\theta - \hat{\theta}(x, y)),$$

(2) *there exist constants $c > 0$ and $C > 0$, such that for all $(x, y) \in (\Omega \times \Omega)\backslash\mathrm{Diag}$,*

$$c \leq \frac{D(x, y)}{d(x, y)} \leq C.$$

Proof. Let $(x, y) \in (\Omega \times \Omega)\backslash\mathrm{Diag}$. Throughout this proof, we will use the following notation:

$$\begin{aligned} &d_1 = d(p, x),\ \ d_2 = d(p, y),\ \ d = d(x, y), \\ &\widetilde{d_1} = \tan d(x, p),\ \ \widetilde{d_2} = \tan d(y, p). \end{aligned} \tag{6.30}$$

By (6.9), we can thus write $\widetilde{\Pi}(\theta, x) = \widetilde{d_1}\cos(\theta - \alpha(x))$ and $\widetilde{\Pi}(\theta, y) = \widetilde{d_2}\cos(\theta - \alpha(y))$, for all $\theta \in \mathbb{R}$. In order to make the calculations clearer, write $\alpha = \theta - \alpha(y)$ and $\alpha_0 = \alpha(x) - \alpha(y)$. With this notation, the proof of Claim (1) is similar to the proof of Claim (1) in Lemma 6.3.

In order to prove Claim (2), it suffices to show that $c \leq \frac{\sqrt{A^2+B^2}}{d} \leq C$, for constants $c > 0$ and $C > 0$ independent of x and y. Recall that A and B are defined as

$$A = \widetilde{d_1}\cos\alpha_0 - \widetilde{d_2} \quad \text{and} \quad B = \widetilde{d_1}\sin\alpha_0, \tag{6.31}$$

where $\alpha_0 = \alpha(x) - \alpha(y)$, see (6.14) and (6.16).

By the spherical law of cosines (6.24), we have

$$\cos d = \cos d_1 \cos d_2 + \sin d_1 \sin d_2 \cos\alpha_0.$$

Since d_1 and d_2 are both strictly smaller than $\frac{\pi}{2}$, $\cos d_1 \cos d_2 \neq 0$, and we obtain

$$-2\tan d_1 \tan d_2 \cos\alpha_0 = 2\left(1 - \frac{\cos d}{\cos d_1 \cos d_2}\right). \tag{6.32}$$

From (6.31), (6.32) and elementary calculation rules for trigonometric functions it follows that

$$A^2 + B^2 = \frac{\cos^2 d_1 + \cos^2 d_2 - 2\cos d \cos d_1 \cos d_2}{\cos^2 d_1 \cos^2 d_2}. \tag{6.33}$$

Recall that $d_1, d_2 \in (0, R]$ where $0 < R < \frac{\pi}{2}$. Therefore $0 < \cos d_1$ and $\cos d_2 < 1$.

Hence, we can derive the following lower bound for A^2+B^2:

$$A^2+B^2 \geq \frac{2\cos d_1 \cos d_2 - 2\cos d \cos d_1 \cos d_2}{\cos^2 d_1 \cos^2 d_2} = \frac{2(1-\cos d)}{\cos d_1 \cos d_2} \geq 2(1-\cos d).$$

This implies that

$$\frac{\sqrt{A^2+B^2}}{d} \geq \sqrt{2}\frac{\sqrt{1-\cos d}}{d}. \tag{6.34}$$

The function $d \mapsto \frac{\sqrt{1-\cos d}}{d}$ is continuous on $(0,\infty)$ and $\lim_{d\to 0^+} \frac{\sqrt{1-\cos d}}{d} = \frac{1}{\sqrt{2}} > 0$. Since $0 < d < 2m < \pi$, it follows that there exists a constant c only depending on R such that $\sqrt{2}\frac{\sqrt{1-\cos d}}{d} \geq c$. This together with (6.34) proves the left-hand inequality in Claim (2). Now, let us prove the right-hand inequality. We define $\rho = d_1 - d_2$, thus by the triangle inequality $0 < |\rho| \leq |d| < \pi$ and therefore $\cos d \leq \cos\rho$. The following calculation only uses the definition of ρ and elementary calculation rules for cos.

$$\begin{aligned}
&\cos^2 d_1 + \cos^2 d_2 - 2\cos d \cos d_1 \cos d_2 \\
=\;& \cos^2(d_2+\rho) + \cos^2 d_2 - 2\cos d \cos(d_2+\rho)\cos d_2 \\
=\;& \frac{1}{2}(\cos(2(d_2+\rho))+1) + \frac{1}{2}(\cos(2d_2)+1) - \cos d(\cos(2d_2+\rho)+\cos\rho) \\
=\;& 1 + \frac{1}{2}(\cos(2(d_2+\rho)) + \cos(2d_2)) - \cos d(\cos(2d_2+\rho)+\cos\rho) \\
=\;& 1 + \cos(2d_2+\rho)\cos\rho - \cos d(\cos(2d_2+\rho)+\cos\rho) \\
=\;& 1 - \cos d\cos\rho + (\cos\rho - \cos d)\cos(2d_2+\rho) \\
\leq\;& 1 - \cos d\cos\rho + (\cos\rho - \cos d) \;\leq\; 2(1-\cos d).
\end{aligned}$$

Note that $2(1-\cos d) \leq d^2$ for $0 < d < 2R < \pi$. Consequently, the estimate

$$\cos^2 d_1 + \cos^2 d_2 - 2\cos d \cos d_1 \cos d_2 \leq d^2 \tag{6.35}$$

follows. Recall that $d_1, d_2 < R$. Set $C = \frac{1}{\cos^4 R}$, then $\frac{1}{\cos^2 d_1 \cos^2 d_2} \leq C$ and hence, by (6.33) and (6.35), we obtain $\frac{\sqrt{A^2+B^2}}{d} \leq C$. □

6.2 HYPERBOLIC N-SPACE

By $\mathbb{H}^n$ denote the hyperbolic n-space and by d the hyperbolic metric on $\mathbb{H}^n$. As in Section 6.1.1, we fix a base point $p \in \mathbb{H}^n$ and identify the tangent plane $T_p\mathbb{H}^n$ with $\mathbb{R}^n$. Now, consider the exponential mapping $\exp_p : \mathbb{R}^n \to \mathbb{H}^n$ at p. Let $V \in G(n,m)$. Then $\exp_p(V)$ is a geodesically convex m-dimensional submanifold of $\mathbb{H}^n$ that is isometric to $\mathbb{H}^m$. Recall that $\mathbb{H}^n$ is a simply connected Riemannian manifold of constant sectional curvature equal to -1. Thus, for all $x \in \mathbb{H}^n$, there exists a unique point $q \in \exp_p(V)$ such that $\mathrm{dist}(x, \exp_p(V)) = d(x,q)$; see Proposition 2.4 in [8]. This point q is called the projection of x onto $\exp_p(V)$ and we denote it by $P_V(x)$. We call the mapping

$$P : G(n,m) \times \mathbb{H}^n \to \mathbb{H}^n$$

defined by $P(V,x) := P_V(x)$, for $x \in \mathbb{H}^n$ and $V \in G(n,m)$, the family of closest-point projections onto hyperbolic m-planes in $\mathbb{H}^n$. Moreover, Proposition 2.4 in [8] implies that the mappings $P_V : \mathbb{H}^n \to \exp_p(V)$ are 1-Lipschitz, and hence $\dim P_V(A) \leq \dim A$, for all $A \subseteq \mathbb{H}^n$. The same proposition also implies that for all $x \in \mathbb{H}^n$ and $V \in G(n,m)$ the geodesic segment $[x, P_V(x)]$ intersects $\exp_p(V)$ orthogonally in the point $P_V(x)$. Therefore, we will refer to $P : G(n,m) \times \mathbb{H}^n \to \mathbb{H}^n$ as the family of orthogonal projections (along geodesics) onto m-planes in $\mathbb{H}^n$.

Consider the Poincaré model of hyperbolic n-space $\mathbb{H}^n$, that is, the metric space $(D^n, d_{\mathcal{P}})$ where $D^n := \{x \in \mathbb{R}^n : |x| < 1\}$ and for all $x, y \in D^n$,

$$d_{\mathcal{P}}(x,y) = 2\,\mathrm{atanh}\left(\frac{|x-y|}{(1 - 2\langle x, y\rangle + |x|^2 + |y|^2)^{\frac{1}{2}}}\right). \tag{6.36}$$

Let Γ be a circle in $\mathbb{R}^n$ that intersect ∂D^n orthogonally. Then $\Gamma \cap D^n$ is a hyperbolic geodesic in the Poincaré model $(D^n, d_{\mathcal{P}})$. The same holds for $L \cap D^n$ for $L \in G(n,1)$. Conversely, every geodesic of hyperbolic space displayed in the Poincaré model is distance minimizing with respect to $d_{\mathcal{P}}$ and is either of the type $\Gamma \cap D^n$ or $L \cap D^n$. Moreover, the Poincaré model is known to be a conformal model of hyperbolic space. This means that the angle in which two curves in hyperbolic n-space intersect equals the Euclidean angle in which their representatives in $(D^n, d_{\mathcal{P}})$ intersect. This makes the Poincaré model a natural choice for studying orthogonal projections of hyperbolic n-space.

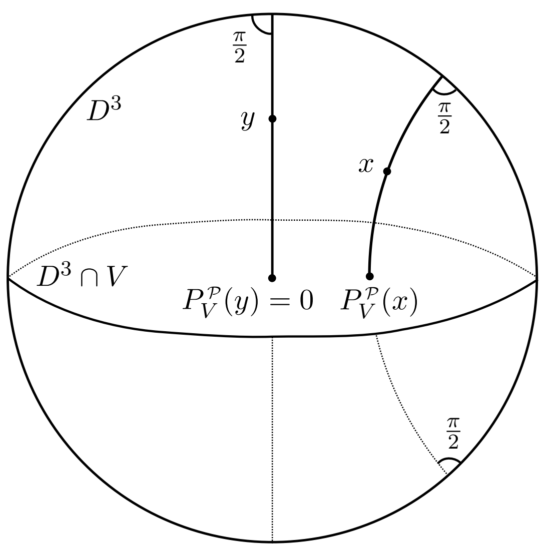

Figure 6.1. The projection $P_V^{\mathcal{P}} : D^3 \to D^3 \cap V$.

Choose 0 to be the representative of the base point $p \in \mathbb{H}^n$ in the model $(D^n, d_{\mathcal{P}})$. This choice is made without loss of generality since $\mathbb{H}^n$ is homogeneous with respect to its group of isometries. Then, for all $V \in G(n,m)$, the hyperbolic m-plane $\exp_p(V)$ corresponds to the m-dimensional disc $V \cap D^n$ in the model $(D^n, d_{\mathcal{P}})$. For each m-plane $V \in G(n,m)$, define $P_V^{\mathcal{P}} : D^n \to V \cap D^n$ to be the closest-point projection onto $V \cap D^n$ with respect to the metric d_p; see Figure 6.1. By conformality of the Poincaré model $(D^n, d_{\mathcal{P}})$, the family $P : G(n,m) \times \mathbb{H}^n \to \mathbb{H}^n$ can be viewed as the family of projections $P^{\mathcal{P}} : G(n,m) \times D^n \to D^n$ defined by $P^{\mathcal{P}}(V,x) = P_V^{\mathcal{P}}(x)$.

Now, consider the mapping $\Psi : D^n \to D^n$, defined by

$$\Psi(x) := \frac{\tanh(\frac{1}{2}\operatorname{atanh}(|x|))}{|x|} x, \tag{6.37}$$

for all $x \in D^n$. Notice that Ψ is a bijection with inverse $\Psi^{-1} : D^n \to D^n$ given by

$$\Psi^{-1}(y) = \frac{\tanh(2\operatorname{atanh}(|y|))}{|y|} y.$$

Moreover, one can check that Ψ maps geodesics $\Gamma \cap D^n$ (where either $\Gamma \in G(n,1)$ or Γ is a circle that intersects ∂D^n orthogonally) to the Euclidean line segment that connects the points $p_1, p_2 \in \partial D^n \cap \Gamma$; see Figure 6.2.

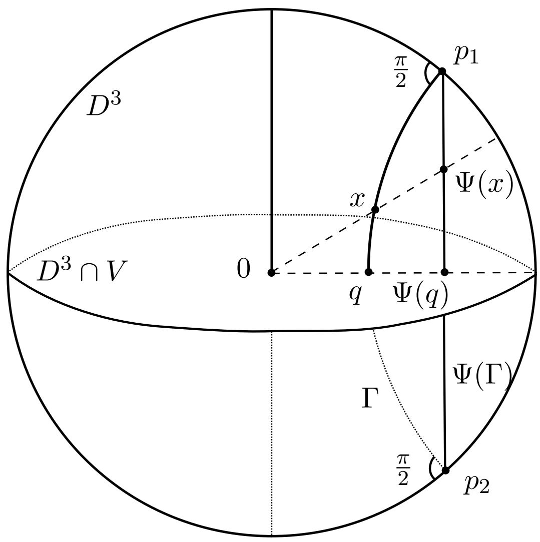

Figure 6.2. The mapping $\Psi : D^3 \to D^3$ where Γ is a geodesic in $(D^3, d_{\mathcal{P}})$.

Notice that the metric space $(D^n, d_{\mathcal{K}})$ where $d_{\mathcal{K}}(x,y) = d_{\mathcal{P}}(\Psi^{-1}(x), \Psi^{-1}(y))$, for all $x, y \in D^n$, is often called the Klein model or the projective model of hyperbolic space; see [6] for details.

As we shall see, the symmetry of Ψ yields the following relation between orthogonal projections in the Poincaré model and Euclidean orthogonal projections:

$$P_V^{\mathcal{P}}(x) = \Psi(P_V^{\mathbb{E}}(\Psi^{-1}(x))), \tag{6.38}$$

for all $V \in G(n,m)$ and $x \in D^n$. To see this, let $x \in D^n$ and $V \in G(n,m)$. By Γ denote the circular arc in D^n that is perpendicular to V and ∂D^n and contains x. Then, by definition, $P_V^{\mathcal{P}}(x)$ is the unique intersection point of V and Γ. Since Γ intersects V orthogonally, the set $\Gamma \cap \partial D^n = \{p_1, p_2\}$ is symmetric under the reflection through V. Thus, the line segment $\Psi(\Gamma)$ connecting the two points p_1 and p_2 intersects V orthogonally; see Figure 6.2. By definition, $\Psi(x)$ is the unique intersection point of Γ with the ray that emerges from the origin and goes through x within D^n. Then, since $\Psi(x) \in \Psi(\Gamma)$, and $\Psi(\Gamma)$ intersects V orthogonally, $P_V^{\mathbb{E}}(x)$ is the point where $\Psi(\Gamma)$ intersects $V \cap D^n$. On the other hand, $\Psi(P_V^{\mathcal{P}}(x))$ is the intersection point of $\Psi(\Gamma)$ and the ray that emerges from the origin and passes through $P_V^{\mathcal{P}}(x)$. However, this intersection point is exactly $P_V^{\mathbb{E}}(\Psi(x))$; see Figure 6.2. This proves (6.38).

The mapping $\Psi : D^n \to D^n$ obviously is a diffeomorphism on $D^n \backslash \{0\}$ and thus locally bi-Lipschitz on $D^n \backslash \{0\}$. Moreover, notice that also the metric $d_{\mathcal{P}}$ is locally bi-Lipschitz to the Euclidean metric on D^n. Hence, the following theorem is a straight-forward consequence of the fact that Theorem 3.11 and Theorem 3.4 hold for Euclidean projections with $L = \infty$; see Remark 3.13.

Theorem 6.7. *For the family $P : G(n,m) \times \mathbb{H}^n \to \mathbb{H}^n$ of orthogonal projections onto m-planes in $\mathbb{H}^n$ and all Borel sets $A \subseteq \mathbb{H}^n$, the following hold.*

(1) *If* $\dim A \leq m$, *then*
   - *(a)* $\dim(P_V A) = \dim A$ *for $\sigma_{n,m}$-a.e. $V \in G(n,m)$,*
   - *(b) For* $0 < \alpha \leq \dim A$,
     $\dim(\{V \in G(n,m) : \dim(P_V A) < \alpha\}) \leq (n-m-1)m + \alpha$.

(2) *If* $\dim A > m$, *then*
   - *(a)* $\mathscr{H}^m(P_V A) > 0$ *for $\sigma_{n,m}$-a.e. $V \in G(n,m)$,*
   - *(b)* $\dim(\{V \in G(n,m) : \mathscr{H}^m(P_V A) = 0\}) \leq (n-m)m + m - \dim A$.

(3) *If* $\dim A > 2m$, *then*
   - *(a)* $P_V A \subseteq V \simeq \mathbb{R}^m$ *has non-empty interior for $\sigma_{n,m}$-a.e. $V \in G(n,m)$,*
   - *(b)* $\dim(\{V \in G(n,m) : (P_V A)^\circ \neq \varnothing\}) \leq (n-m)m + 2m - \dim A$.

Moreover, a set $\widetilde{A} \subseteq \mathbb{H}^n$ with $\mathscr{H}^m(A) < \infty$ is purely m-unrectifiable if and only if $\mathscr{H}^m(P_V(\widetilde{A})) = 0$ for $\sigma_{n,m}$-a.e. $V \in G(n,m)$.

The rest of this section is concerned with the proof of differentiable transversality for the family of orthogonal projections onto m-planes in $\mathbb{H}^n$. For this, let $\Omega \subset D^n$ be a compact ball with center 0 and radius $0 < R < 1$ and let $\varphi : \mathrm{Mat}_{(n-m)\times m}(\mathbb{R}) \to G(n,m)$

be the local parameterization of $G(n,m)$ as introduced in Section 2.3. Recall that for all $T \in \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$ by $e_1^T, \ldots, e_m^T$ we denote an orthonormal basis of $\varphi(T) \in G(n,m)$ that varies smoothly in T. Define the family of abstract orthogonal projections onto m-planes in the Poincaré model $(D^n, d_{\mathcal{P}})$ to be $\Pi^{\mathcal{P}} : \mathrm{Mat}_{(n-m)\times m}(\mathbb{R}) \times \Omega \to D^m$ where

$$\Pi^{\mathcal{P}}(T,x) := \sum_{i=1}^{m} \langle P^{\mathcal{P}}(T,x), e_i^T \rangle w_i, \tag{6.39}$$

for all $x \in \Omega$ where $w_1, \ldots, w_m$ is the standard basis of $\mathbb{R}^m$.

Theorem 6.8. *The family $\Pi^{\mathcal{P}} : \mathrm{Mat}_{(n-m)\times m}(\mathbb{R}) \times \Omega \to D^m$ of abstract orthogonal projections onto m-planes in the Poincaré model satisfies differentiable transversality with $L = 2$ and $\delta = 0$. Moreover, $\Pi^{\mathcal{P}} : \mathrm{Mat}_{(n-m)\times m}(\mathbb{R}) \times \Omega \to D^m$ is a C^2-mapping and thus Theorem 3.14 applies.*

In order to prove Theorem 6.8, we first prove a sequence of technical lemmas. To this end, let $F : D^n \to D^n$ be given by

$$F(x) = \rho(|x|)x,$$

where $\rho : [0,1) \to (0,\infty)$. Moreover, denote the matrix of the identity $\mathbb{R}^n \to \mathbb{R}^n$ by I_n.

Lemma 6.9. *For F and ρ as above, we assume that*

(a) ρ *is of class* C^2 *on* $(0,1)$,

(b) ρ, $\dot\rho$ *and* $\ddot\rho$ *have a continuous extension to* $[0,1)$,

(c) ρ *is non-decreasing and* $\dot\rho(0) = 0$.

Then, the following hold.

(i) F *is* C^1 *and* $\mathrm{D}F(0) = \rho(0)\,\mathrm{I}_n$,

(ii) $\det(\mathrm{D}F(x)) > 0$ *for all* $x \in D^n$,

(iii) F *is of class* C^2.

The proof of Lemma 6.9 will show that parts (i) and (ii) only require ρ to be C^1 and non-decreasing. The conditions $\rho \in C^2$ and $\dot\rho(0) = 0$ are only needed for part (iii).

Proof of Lemma 6.9. Notice that F is of class C^1 on $D^n \setminus \{0\}$ by definition. In order to show that the differential at zero exists and equals $\rho(0)\,\mathrm{I}_n$, it suffices to check

$$\lim_{|x|\to 0} \frac{F(x) - F(0) - \rho(0)(x-0)}{|x|} = 0. \tag{6.40}$$

Since $F(0) = 0$, equation (6.40) is equivalent to

$$\lim_{|x|\to 0} \frac{|F(x) - \rho(0)(x-0)|}{|x|} = 0. \tag{6.41}$$

Consider the following calculation

$$\frac{|F(x)-\rho(0)(x-0)|}{|x|}=\frac{|\rho(|x|)x-\rho(0)x|}{|x|}=\frac{|\rho(|x|)-\rho(0)|\;|x|}{|x|}=|\rho(|x|)-\rho(0)|.$$

Thus, by continuity of ρ, (6.41) follows and consequently $\mathrm{D}F(0)=\rho(0)\,\mathrm{I}_n$.

Now, let $x\in D^n$ so that $|x|>0$. Then, by the chain rule, F is continuously differentiable in x and the differential is the $(n\times n)$-matrix

$$\mathrm{D}F(x)=\dot{\rho}(|x|)\frac{1}{|x|}\,[x_ix_j]_{i,j=1}^n+\rho(|x|)\mathrm{I}_n. \tag{6.42}$$

By continuity of ρ,

$$\lim_{x\to 0}\rho(|x|)\mathrm{I}_n=\rho(0)\mathrm{I}_n=\mathrm{D}F(0).$$

Furthermore, since $\frac{x_ix_j}{|x|}$ is bounded and $\dot{\rho}$ is continuous, it follows that

$$\lim_{x\to 0}\dot{\rho}(|x|)\frac{1}{|x|}[x_ix_j]_{i,j=1}^n=0. \tag{6.43}$$

Thus, $\mathrm{D}F$ is continuous in zero and hence F is of class C^1 in D^n. This proves (i).

Now, we prove (ii). First, let $x=0$. As in the proof of (i), we have $\mathrm{D}F(0)=\rho(0)\,\mathrm{I}_n$, and thus $\det \mathrm{D}F(0)=\rho(0)^n>0$. (Recall that we have chosen ρ to be strictly positive.) Now, let $x_a:=(a,0,\dots,0)^\mathsf{T}$ for some $0<a<1$. Then, by (6.42),

$$\mathrm{D}F(x_a)=\dot{\rho}(a)\;\frac{1}{a}\begin{pmatrix} a^2 & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & & & \vdots\\ 0 & & \dots & 0\end{pmatrix}+\rho(a)\;\mathrm{I}_n$$

and thus,

$$\mathrm{D}F(x_a)=\begin{pmatrix} \rho(a)+a\dot{\rho}(a) & & \\ & \rho(a) & & 0\\ & 0 & \ddots & \\ & & & \rho(a)\end{pmatrix}. \tag{6.44}$$

Recall that ρ is assumed to be non-decreasing. Thus, (6.44) immediately implies,

$$\det(\mathrm{D}F(x_a))=(\rho(a)+a\dot{\rho}(a))\rho(a)^{n-1}>0.$$

Now, let $y\in D^n\backslash\{0\}$ and set $a=\|y\|$ and $x=x_a$. Then, there exists $A\in\mathrm{SO}(n)$ such that $y=Ax$. Since $A\in\mathrm{SO}(n)$, by the chain rule

$$\mathrm{D}(F\circ A)(x)=\mathrm{D}F(Ax)\;\mathrm{D}A(x)=\mathrm{D}F(y)\;A$$

On the other hand, by definition of F

$$F \circ A(x) = \rho(|Ax|)Ax = \rho(|x|)Ax = A\rho(|x|)x = A \circ F(x),$$

and thus

$$\mathrm{D}(F \circ A)(x) = \mathrm{D}(A \circ F)(x) = (\mathrm{D}A)(F(x))\ \mathrm{D}F(x) = A\ \mathrm{D}F(x).$$

Hence, it follows that

$$\det(\mathrm{D}F(y)) = \det(A)\det(\mathrm{D}F(x))\det(A^{-1}) = \det(\mathrm{D}F(x)) > 0.$$

This proves (ii).

Thus, we are left to show that $\mathrm{D}F : D^n \to \mathbb{R}^{n\times n},\ x \mapsto \mathrm{D}F(x)$, is of class C^1, i.e., that each entry $m_{i,j}(x)$ of $\mathrm{D}F(x)$, $i, j \in \{1,\dots,n\}$, is continuously differentiable in D^n.

By (6.40) and (6.42), it follows that

$$\begin{aligned} m_{i,j}(x) &= \dot\rho(|x|)\frac{1}{|x|}x_i x_j + \rho(|x|)\delta_{ij}\ , \quad \text{for } x \in \mathbb{R}^n\backslash\{0\} \\ m_{i,j}(0) &= \rho(0)\delta_{ij} \end{aligned}$$

Let $l, i, j \in \{1,\dots,n\}$. By the chain rule, $m_{i,j}$ is of class C^1 in $D^n\backslash\{0\}$ and for all $x \in D^n\backslash\{0\}$,

$$\begin{aligned} \frac{\partial}{\partial x_l}m_{i,j}(x) = \ddot\rho(|x|)\frac{1}{|x|^2}x_l x_i x_j \\ - \dot\rho(|x|)\left(\frac{1}{|x|^3}x_l x_i x_j - \frac{1}{|x|}(\delta_{li}x_j + \delta_{lj}x_i) - \delta_{ij}\frac{1}{|x|}x_l\right). \end{aligned} \tag{6.45}$$

Moreover, by the definition of partial derivatives,

$$\begin{aligned} \frac{\partial}{\partial x_l}m_{i,j}(0) &= \lim_{h\to 0}\frac{1}{|h|}\left(m_{i,j}((0,\dots,0,h,0,\dots,0)) - m_{i,j}(0)\right) \\ &= \lim_{h\to 0}\frac{1}{|h|}\left(\dot\rho(h)\frac{1}{|h|}h^2\delta_{il}\delta_{jl} + \rho(h)\delta_{ij} - \rho(0)\delta_{ij}\right) \\ &= \lim_{h\to 0}\left(\dot\rho(|h|)\frac{h^2}{|h|^2}\right)\delta_{il}\delta_{jl} + \delta_{ij}\lim_{h\to 0}\left(\frac{\rho(|h| - \rho(0)}{|h|}\right) \\ &= \lim_{h\to 0}\left(\dot\rho(|h|)\frac{h^2}{|h|^2}\right)\delta_{il}\delta_{jl} + \delta_{ij}\dot\rho(0) \end{aligned}$$

Since $\dot\rho$ is continuous, the assumption that $\dot\rho(0) = 0$ yields

$$\frac{\partial}{\partial x_l}m_{i,j}(0) = 0. \tag{6.46}$$

Analogously, by using the continuity of $\dot\rho$ and $\ddot\rho$, as well as the assumption that $\dot\rho(0) = 0$, if follows that

$$\lim_{x\to 0} \frac{\partial}{\partial x_l} m_{i,j}(x) = 0 = \frac{\partial}{\partial x_l} m_{i,j}(0)$$

Thus, $m_{i,j}$ is continuously differentiable in D^n for all $i, j \in \{1, \ldots, n\}$. This proves (iii). $\square$

Lemma 6.10. *Under the assumptions of Lemma 6.9,*

$$|\mathrm{D}F(x) - \mathrm{D}F(y)| = \mathcal{O}(|x - y|),$$

for all $x, y \in D^n$, where $\mathcal{O}$ denotes the Bachmann-Landau symbol (big O).

Proof of Lemma 6.10. Recall that $\mathrm{D}F\colon D^n \to \mathbb{R}^{n\times n},\ x \mapsto \mathrm{D}F(x)$ is a C^1-mapping. As before, let $m_{i,j}(x)$ for $i, j \in \{1, \ldots, n\}$ and $x \in D^n$ be defined by $[m_{i,j}(x)]_{i,j=1}^n := \mathrm{D}F(x)$. Thus, $m_{i,j} : D^n \to \mathbb{R}$ is a C^1-mapping. By the higher dimensional version of Taylors theorem with qualitative estimate for the remainder term, we obtain

$$m_{i,j}(x) = m_{i,j}(y) + \mathcal{O}(|x - y|)$$

for all $x, y \in D^n$. Thus,

$$|\mathrm{D}F(x) - \mathrm{D}F(y)| = |[m_{i,j}(x)]_{i,j=1}^n - [m_{i,j}(y)]_{i,j=1}^n| = \mathcal{O}(|x - y|).$$

$\square$

We now want to apply the lemmas above to a specific function $\psi : [0, 1) \to (0, \infty)$.

Lemma 6.11. *The function $\psi : \mathbb{R} \to \mathbb{R}$ defined by*

$$\psi(r) = \frac{\tanh(\frac{1}{2}\mathrm{atanh}(r))}{r}$$

is a C^∞-mapping and its restriction to $[0, 1)$ satisfies the assumptions of Lemma 6.9.

Notice that $\Psi(x) = \psi(|x|)x$, for all $x \in D^n$; see (6.37). Thus, by Lemma 6.11, it follows that Lemma 6.9 holds for $F = \Psi$ (for all $n \geq 2$).

Proof. From the Taylor decompositions of the hyperbolic functions $\tanh : \mathbb{R} \to \mathbb{R}$ and $\mathrm{atanh} : \mathbb{R} \to \mathbb{R}$, one easily deduces

$$\tanh(\tfrac{1}{2}\mathrm{atanh}(r)) = \frac{r}{2} + \frac{r^3}{8} + \frac{r^5}{16} + \mathcal{O}(r^7).$$

Therefore, it follows that ψ is well-defined and C^∞. Moreover, it follows that $\dot\psi(0) = 0$ and hence the restriction of $\psi : [0, 1) \to [0, \infty)$ satisfies the assumptions of Lemma 6.9. $\square$

Towards the proof of Theorem 6.8, define

$$\Phi^{\mathcal{P}}(T,x,y) := \frac{\Pi^{\mathcal{P}}(T,x) - \Pi^{\mathcal{P}}(T,x)}{d_{\mathcal{P}}(x,y)}$$

and

$$\widetilde{\Phi}^{\mathcal{P}}(T,x,y) := \Pi^{\mathcal{P}}(T,x) - \Pi^{\mathcal{P}}(T,x),$$

for all $(x,y) \in (\Omega \times \Omega)\backslash \mathrm{Diag}$ and $T \in \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$; compare (3.6). Define $\widetilde{\Phi}^{\mathbb{E}}(T,x,y)$ analogously in terms $P^{\mathbb{E}}$. Then, trivially,

$$\Phi^{\mathcal{P}}(T,x,y) = \frac{\widetilde{\Phi}^{\mathcal{P}}(T,x,y)}{d_{\mathcal{P}}(x,y)} \tag{6.47}$$

and,

$$\Phi^{\mathbb{E}}(T,x,y) = \frac{\widetilde{\Phi}^{\mathbb{E}}(T,x,y)}{|x-y|}.$$

for all $(x,y) \in (\Omega \times \Omega)\backslash \mathrm{Diag}$ and $T \in \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$.

Proof of Theorem 6.8. Recall from (6.39) that

$$\Pi^{\mathcal{P}}(T,x) := \sum_{i=1}^{m} \langle P^{\mathcal{P}}(T,x), e_i^T \rangle w_i,$$

and from (6.38) that $P_V^{\mathcal{P}}(x) = \Psi(P_V^{\mathbb{E}}(\Psi^{-1}(x)))$. Moreover, recall that $\Psi : D^n \to D^n$ is a C^2-mapping (Lemma 6.9 and Lemma 6.11), and that $P^{\mathbb{E}} : G(n,m) \times \mathbb{R}^n \to \mathbb{R}^n$ is C^∞ in the first variable and linear in the second variable. Consequently, $P^{\mathbb{E}}$ is a C^∞-mapping on $G(n,m) \times \mathbb{R}^n$, and we may conclude that $\Pi^{\mathcal{P}} : \mathrm{Mat}_{(n-m)\times m}(\mathbb{R}) \times \Omega \to \mathbb{R}^m$ is a C^2-mapping. In particular, $\Pi^{\mathcal{P}}$ satisfies condition (a) of Definition 3.9 for $L = 2$ and $\delta = 0$. Condition (c) can be proven analogously for the same values of L and δ.

Thus, we are left to prove condition (b). For this, recall that for all $T \in \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$, the abstract projection $\Pi(T, \cdot)$ is the projection $P(\varphi(T), \cdot)$ up to identification of $\varphi(T) \in G(n,m)$ with $\mathbb{R}^m$ by a linear isometry; see (6.39). Notice that by the symmetry of Ψ (see (6.37) and Figure 6.2), for all $V \in G(n,m)$ and every linear isometry $i : V \to \mathbb{R}^m$, we have $\Psi \circ i = i \circ \Psi$. Thus, by (6.38), it follows that,

$$\Pi^{\mathcal{P}} = \Psi \circ \Pi^{\mathbb{E}} \circ \Psi^{-1}, \tag{6.48}$$

in the sense that $\Pi^{\mathcal{P}}(T,x) = \Psi(\Pi^{\mathbb{E}}(T, \Psi^{-1}(x))$ for all $x \in \Omega$ and $T \in \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$.

We claim that in order to establish condition (b) of Definition 3.9 for $\Pi^{\mathcal{P}}$, it suffices to establish the following variant of condition (b):

(b') there exists $\widetilde{C}_{\mathcal{P}} > 0$ such that for all $(x,y) \in (\Omega\times\Omega)\backslash\mathrm{Diag}$ and $T \in \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$, whenever $|\widetilde{\Phi}^{\mathcal{P}}(T,x,y)| < \widetilde{C}_{\mathcal{P}}$, then

$$|\det(\mathrm{D}_T\widetilde{\Phi}^{\mathcal{P}}(T,x,y)\left(\mathrm{D}_T\widetilde{\Phi}^{\mathcal{P}}(T,x,y)\right)^{\mathsf{T}})| > \widetilde{C}_{\mathcal{P}}^2. \tag{6.49}$$

To see this, assume that (b') holds with constant $\widetilde{C}_{\mathcal{P}} > 0$. Set $C_{\mathcal{P}} := \frac{\widetilde{C}_{\mathcal{P}}}{\mathrm{diam}_{\mathcal{P}}\,\Omega}$ and let $(x,y) \in (\Omega\times\Omega)\backslash\mathrm{Diag}$ and $T \in \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$ such that $|\Phi^{\mathcal{P}}(T,x,y)| < C_{\mathcal{P}}$. Then, by (6.47), $|\widetilde{\Phi}^{\mathcal{P}}(T,x,y)| < \widetilde{C}_{\mathcal{P}}$. Thus, and by (b') and the linearity of the differential D_T, it follows that $|\det(\mathrm{D}_T\Phi^{\mathcal{P}}(T,x,y)\,(\mathrm{D}_T\Phi^{\mathcal{P}}(T,x,y))^{\mathsf{T}})| > \frac{\widetilde{C}_{\mathcal{P}}^2}{d_{\mathcal{P}}(x,y)^2} \geq C_{\mathcal{P}}^2$. This proves that (b') implies (b) for the family $\Phi^{\mathcal{P}}$.

Now, we prove that (b') holds for the family of abstract projections $\Pi^{\mathcal{P}}$ by applying the fact that (b') holds for the family of abstract Euclidean projections $\Pi^{\mathbb{E}}$ with constant $\widetilde{C}_{\mathbb{E}} > 0$. By the chain rule and (6.48), it follows that

$$\mathrm{D}_T\Pi^{\mathcal{P}}(T,x) = \mathrm{D}\Psi(\Pi^{\mathbb{E}}(T,x))\;\mathrm{D}_T\Pi^{\mathbb{E}}(T,\Psi^{-1}(x))$$

for all $x \in \Omega$ and $T \in \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$.

Let $(x,y) \in (\Omega\times\Omega)\backslash\mathrm{Diag}$ and $T \in \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$. For the sake of readability of the upcoming calculation, we will slightly abuse notation and abbreviate the preimages of the points x and y under Ψ by $u = \Psi^{-1}(x)$ and $v = \Psi^{-1}(y)$. And we may write,

$$\begin{aligned}
\mathrm{D}_T\widetilde{\Phi}^{\mathcal{P}}(T,x,y) &= \mathrm{D}\Psi(\Pi^{\mathbb{E}}(T,u))\;\mathrm{D}_T\Pi^{\mathbb{E}}(T,u) - \mathrm{D}\Psi(\Pi^{\mathbb{E}}(T,v))\;\mathrm{D}_T\Pi^{\mathbb{E}}(T,v)\\
&= \mathrm{D}\Psi(\Pi^{\mathbb{E}}(T,u))\;\left[\mathrm{D}_T\Pi^{\mathbb{E}}(T,u) - \mathrm{D}_T\Pi^{\mathbb{E}}(T,v)\right]\\
&\quad - \left[\mathrm{D}\Psi(\Pi^{\mathbb{E}}(T,v)) - \mathrm{D}\Psi(\Pi^{\mathbb{E}}(T,u))\right]\;\mathrm{D}_T\Pi^{\mathbb{E}}(T,v)\\
&= \mathrm{D}\Psi(\Pi^{\mathbb{E}}(T,u))\;\mathrm{D}_T\widetilde{\Phi}(T,u,v) + \Delta(T,x,y)\;\mathrm{D}_T\Pi^{\mathbb{E}}(T,v)
\end{aligned}$$

where $\Delta(T,x,y) := \mathrm{D}\Psi(\Pi^{\mathbb{E}}(T,v)) - \mathrm{D}\Psi(\Pi^{\mathbb{E}}(T,u))$.

Thus, it follows that

$$\begin{aligned}
&\mathrm{D}_T\widetilde{\Phi}^{\mathcal{P}}(T,x,y)(\mathrm{D}_T\widetilde{\Phi}^{\mathcal{P}}(T,x,y))^{\mathsf{T}}\\
&= \mathrm{D}\Psi(\Pi^{\mathbb{E}}(T,u))\left[\mathrm{D}_T\widetilde{\Phi}^{\mathbb{E}}(T,u,v)\;(\mathrm{D}_T\widetilde{\Phi}^{\mathbb{E}}(T,u,v))^{\mathsf{T}}\right](\mathrm{D}\Psi(\Pi(T,u)))^{\mathsf{T}}\\
&\quad + \widetilde{\Delta}(T,x,y),
\end{aligned} \tag{6.50}$$

where

$$\begin{aligned}
\widetilde{\Delta}(T,x,y) \;:=\; &\mathrm{D}\Psi^{\mathcal{P}}(\Pi(T,u))\;\mathrm{D}_T\widetilde{\Phi}^{\mathbb{E}}(T,u,v)\;(\mathrm{D}_T\Pi(T,u))^{\mathsf{T}}\;(\Delta(T,x,y))^{\mathsf{T}}\\
&+ \mathrm{D}\Psi(\Pi^{\mathbb{E}}(T,v))\;\Delta(T,x,y)\;(\mathrm{D}_T\widetilde{\Phi}^{\mathbb{E}}(T,u,v))^{\mathsf{T}}\;(\mathrm{D}\Psi(\Pi^{\mathbb{E}}(T,u)))^{\mathsf{T}}\\
&+ \mathrm{D}\Psi(\Pi^{\mathbb{E}}(T,v))\;\Delta(T,x,y)\;(\mathrm{D}_T\Pi^{\mathbb{E}}(T,v))^{\mathsf{T}}\;(\Delta(T,x,y))^{\mathsf{T}}.
\end{aligned}$$

By Lemma 6.11, Lemma 6.10 applies for Ψ. Therefore, it follows that $\Delta(T,x,y) = \mathcal{O}(|\Pi^{\mathbb{E}}(T,u) - \Pi^{\mathbb{E}}(T,v)|)$, and hence, $\widetilde{\Delta}(T,x,y) = \mathcal{O}(|\Pi^{\mathbb{E}}(T,u) - \Pi^{\mathbb{E}}(T,v)|)$. Recall that we write $\widetilde{\Phi}^{\mathbb{E}}(T,x,y) = \Pi^{\mathbb{E}}(T,u) - \Pi^{\mathbb{E}}(T,v)$. Thus, this yields

$$\widetilde{\Delta}(T,x,y) = \mathcal{O}(|\widetilde{\Phi}^{\mathbb{E}}(T,x,y)|). \tag{6.51}$$

Furthermore, recall that

- The determinant of a matrix is a smooth function in the entries of the matrix,
- $\mathrm{D}\Psi(q) > 0$ on $q \in D^m$ (see Lemma 6.9),
- Ω was chosen to be a closed ball with center 0 in D^n. Therefore, also $\Psi^{-1}(\Omega)$ is a closed ball with center 0 in D^n, and hence, there exists a compact set $\Omega' \subset D^m$ such that $\Pi^{\mathbb{E}}(T, \Psi^{-1}(\Omega)) = \Omega'$ for all $T \in \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$.
- $G(n,m)$ is compact.

In conclusion, there exists a constant $M > 0$ such that

$$\left(\det\left[\mathrm{D}\Psi(\Pi^{\mathbb{E}}(T,u))\right]\right)^2 = \left(\det\left[\mathrm{D}\Psi(\Pi^{\mathbb{E}}(T,\Psi^{-1}(x)))\right]\right)^2 > M \tag{6.52}$$

for all $x \in \Omega$ and $T \in \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$.

Then, since (b) and thus (b') hold for $\Pi^{\mathbb{E}}$, it follows that

$$\left|\det\left[(\mathrm{D}_T\widetilde{\Phi}^{\mathbb{E}}(T,u,v))\,(\mathrm{D}_T\widetilde{\Phi}^{\mathbb{E}}(T,u,v))^{\mathsf{T}}\right]\right| > \widetilde{C}_{\mathbb{E}} \tag{6.53}$$

for all $x, y \in \Omega$ and $T \in \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$ satisfying $\left|\widetilde{\Phi}^{\mathbb{E}}(T,u,v))^{\mathsf{T}}\right| < \widetilde{C}_{\mathbb{E}}$. Hence, (6.50) yields

$$\begin{aligned}
&\det\left[\mathrm{D}_T\widetilde{\Phi}^{\mathcal{P}}(T,x,y)(\mathrm{D}_T\widetilde{\Phi}^{\mathcal{P}}(T,x,y))^{\mathsf{T}} - \widetilde{\Delta}(T,x,y),\right]\\
&= \det\left[\mathrm{D}\Psi(\Pi^{\mathbb{E}}(T,u))\left(\mathrm{D}_T\widetilde{\Phi}^{\mathbb{E}}(T,u,v)\ (\mathrm{D}_T\widetilde{\Phi}^{\mathbb{E}}(T,u,v))^{\mathsf{T}}\right)(\mathrm{D}\Psi(\Pi(T,u)))^{\mathsf{T}}\right]\\
&= \det\left[\mathrm{D}\Psi(\Pi^{\mathbb{E}}(T,u))\right]^2\ \det\left[\mathrm{D}_T\widetilde{\Phi}^{\mathbb{E}}(T,u,v)\ (\mathrm{D}_T\widetilde{\Phi}^{\mathbb{E}}(T,u,v))^{\mathsf{T}}\right]\\
&\geq M\widetilde{C}_{\mathbb{E}}
\end{aligned} \tag{6.54}$$

for all $x, y \in \Omega$ and $T \in \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$ satisfying $\left|\widetilde{\Phi}^{\mathbb{E}}(T,u,v))^{\mathsf{T}}\right| < \widetilde{C}_{\mathbb{E}}$. Then, again using the fact that the determinant of a matrix is a smooth mapping in the entries of the matrix, as well as (6.51), we may choose $c > 0$ such that

$$\begin{aligned}
&\left|\det\left[\mathrm{D}_T\widetilde{\Phi}^{\mathcal{P}}(T,x,y)(\mathrm{D}_T\widetilde{\Phi}^{\mathcal{P}}(T,x,y))^{\mathsf{T}} - \widetilde{\Delta}(T,x,y)\right]\right.\\
&\qquad\left. - \det\left[\mathrm{D}_T\widetilde{\Phi}^{\mathcal{P}}(T,x,y)(\mathrm{D}_T\widetilde{\Phi}^{\mathcal{P}}(T,x,y))^{\mathsf{T}}\right]\right| < \frac{M\widetilde{C}_{\mathbb{E}}}{2}
\end{aligned} \tag{6.55}$$

for all $x, y \in \Omega$ and $T \in \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$ satisfying $\left|\widetilde{\Phi}^{\mathbb{E}}(T,u,v))^{\mathsf{T}}\right| < c$.

By Lemma 6.11, Ψ is a local diffeomorphism on D^n and hence a bi-Lipschitz mapping. Moreover, recall that by definition, $\widetilde{\Phi}^{\mathbb{E}}(T,x,y) = \Pi^{\mathbb{E}}(T,u) - \Pi^{\mathbb{E}}(T,v)$ for all $x, y \in \Omega$ and $T \in \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$. Therefore, by (6.48), we may choose a sufficiently small constant $\widetilde{C}_{\mathcal{P}} > 0$ such that whenever $|\widetilde{\Phi}\mathcal{P}(T,x,y)| < \widetilde{C}_{\mathcal{P}}$ then

$$\left|\widetilde{\Phi}\,\mathbb{E}(T,u,v)\right| < \min\{c, \widetilde{C}_{\mathbb{E}}\}.$$

Now, for all $x, y \in \Omega$ and $T \in \mathrm{Mat}_{(n-m)\times m}(\mathbb{R})$ satisfying $|\widetilde{\Phi}\mathcal{P}(T,x,y)| < \widetilde{C}_{\mathcal{P}}$, by the choice of $\widetilde{C}_{\mathcal{P}}$, equations (6.54) and (6.55) hold for x, y, T and thus

$$\left|\det\left[\mathrm{D}_T\widetilde{\Phi}^{\mathcal{P}}(T,x,y)(\mathrm{D}_T\widetilde{\Phi}^{\mathcal{P}}(T,x,y))^{\mathsf{T}}\right]\right| > \frac{M\widetilde{C}_{\mathbb{E}}}{2}.$$

We may without loss of generality assume that $\widetilde{C}_{\mathcal{P}} < \frac{M\widetilde{C}_{\mathbb{E}}}{2}$ and hence condition (b') holds for the family $\Pi^{\mathcal{P}} : \mathrm{Mat}_{(n-m)\times m}(\mathbb{R}) \times \Omega \to \mathbb{R}^m$ of abstract projections in the Poincaré model. $\square$

REFERENCES

[1] Z. M. Balogh, E. Durand-Cartagena, K. Fässler, P. Mattila, and J. T. Tyson. The effect of projections on dimension in the Heisenberg group. *Rev. Mat. Iberoam.*, 29(2):381–432, 2013.

[2] Z. M. Balogh, K. Fässler, P. Mattila, and J. T. Tyson. Projection and slicing theorems in Heisenberg groups. *Adv. Math.*, 231(2):569–604, 2012.

[3] Z. M. Balogh and A. Iseli. Marstrand type projection theorems for normed spaces. To appear in *Journal of Fractal Geometry.*
Preprint available at: `https://arxiv.org/abs/1802.10563`.

[4] Z. M. Balogh and A. Iseli. Dimensions of projections of sets on Riemannian surfaces of constant curvature. *Proc. Amer. Math. Soc.*, 144(7):2939–2951, 2016.

[5] D. Bate, M. Csörnyei, and B. Wilson. The Besicovitch–Federer projection theorem is false in every infinite-dimensional Banach space. *Israel J. Math.*, 220(1):175–188, 2017.

[6] R. Benedetti and C. Petronio. *Lectures on hyperbolic geometry.* Universitext. Springer-Verlag, Berlin, 1992.

[7] A. S. Besicovitch. On the fundamental geometrical properties of linearly measurable plane sets of points (III). *Math. Ann.*, 116(1):349–357, 1939.

[8] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[9] C. Chen. Restricted families of projections and random subspaces. Preprint available at: `https://arxiv.org/abs/1706.03456`.

[10] M. P. a. do Carmo. *Riemannian geometry.* Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty.

[11] G. Edgar. *Measure, topology, and fractal geometry.* Undergraduate Texts in Mathematics. Springer, New York, second edition, 2008.

[12] K. Falconer. *Fractal geometry.* John Wiley & Sons, Ltd., Chichester, third edition, 2014. Mathematical foundations and applications.

[13] K. J. Falconer. Hausdorff dimension and the exceptional set of projections. *Mathematika*, 29(1):109–115, 1982.

[14] K. J. Falconer and J. D. Howroyd. Projection theorems for box and packing dimensions. *Math. Proc. Cambridge Philos. Soc.*, 119(2):287–295, 1996.

[15] K. J. Falconer and J. D. Howroyd. Packing dimensions of projections and dimension profiles. *Math. Proc. Cambridge Philos. Soc.*, 121(2):269–286, 1997.

[16] K. Fässler and T. Orponen. On restricted families of projections in $\mathbb{R}^3$. *Proc. Lond. Math. Soc. (3)*, 109(2):353–381, 2014.

[17] H. Federer. The (φ, k) rectifiable subsets of n-space. *Trans. Amer. Soc.*, 62:114–192, 1947.

[18] H. Federer. *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[19] R. Hovila. Transversality of isotropic projections, unrectifiability, and Heisenberg groups. *Rev. Mat. Iberoam.*, 30(2):463–476, 2014.

[20] R. Hovila, E. Järvenpää, M. Järvenpää, and F. Ledrappier. Besicovitch-Federer projection theorem and geodesic flows on Riemann surfaces. *Geom. Dedicata*, 161:51–61, 2012.

[21] A. Iseli and K. Wildrick. Iterated function system quasiarcs. *Conform. Geom. Dyn.*, 21:78–100, 2017.

[22] E. Järvenpää, M. Järvenpää, F. Ledrappier, and M. Leikas. One-dimensional families of projections. *Nonlinearity*, 21(3):453–463, 2008.

[23] M. Järvenpää. On the upper Minkowski dimension, the packing dimension, and orthogonal projections. *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes*, (99):34, 1994.

[24] R. Kaufman. On Hausdorff dimension of projections. *Mathematika*, 15:153–155, 1968.

[25] R. Kaufman and P. Mattila. Hausdorff dimension and exceptional sets of linear transformations. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 1(2):387–392, 1975.

[26] J. E. López, C. G. Moreira, S. A. Romana Ibarra, and W. L. Silva. A generalization of Marstrand's theorem and a geometric application.
Preprint available at: `https://arxiv.org/abs/1611.09965`.

[27] J. M. Marstrand. Some fundamental geometrical properties of plane sets of fractional dimensions. *Proc. London Math. Soc. (3)*, 4:257–302, 1954.

[28] P. Mattila. Hausdorff dimension, projections, intersections, and Besicovitch sets. Preprint available at: `https://arxiv.org/abs/1712.09199`.

[29] P. Mattila. Hausdorff dimension, orthogonal projections and intersections with planes. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 1(2):227–244, 1975.

[30] P. Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.

[31] P. Mattila. Hausdorff dimension, projections, and the Fourier transform. *Publ. Mat.*, 48(1):3–48, 2004.

[32] P. Mattila. *Fourier analysis and Hausdorff dimension*, volume 150 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2015.

[33] T. Orponen and L. Venieri. Improved bounds for restricted families of projections to planes in $\mathbb{R}^3$. Preprint available at: `https://arxiv.org/abs/1711.08934`.

[34] Y. Peres and W. Schlag. Smoothness of projections, Bernoulli convolutions, and the dimension of exceptions. *Duke Math. J.*, 102(2):193–251, 2000.

[35] B. Solomyak. Measure and dimension for some fractal families. *Math. Proc. Cambridge Philos. Soc.*, 124(3):531–546, 1998.

ERKLÄRUNG

gemäss Art. 28 Abs. 2 RSL 05

Name/Vorname:	Iseli Annina
Matrikelnummer:	08-114-274
Studiengang:	Mathematik, Dissertation
Titel der Arbeit:	Dimension and projections in normed spaces and Riemannian manifolds
Leiter der Arbeit:	Prof. Dr. Zoltán Balogh

Ich erkläre hiermit, dass ich diese Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen benutzt habe. Alle Stellen, die wörtlich oder sinngemäss aus Quellen entnommen wurden, habe ich als solche gekennzeichnet. Mir ist bekannt, dass andernfalls der Senat gemäss Artikel 36 Absatz 1 Buchstabe r des Gesetzes vom 5. September 1996 über die Universität zum Entzug des auf Grund dieser Arbeit verliehenen Titels berechtigt ist. Ich gewähre hiermit Einsicht in diese Arbeit.

Bern, 27. April 2018 Annina Iseli